Incremental Issuance in a Model of Risky Debt with Proportional Issuance Costs

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Abstract

We investigate the pricing of corporate debt with and without commitment in a dynamic capital structure model with lognormally distributed earnings in which a firm can issue additional debt. Consistent with practice, it is optimal for a firm to only issue debt infrequently and in a lumpy fashion. We provide a non-smooth Markov perfect equilibrium solution as an alternative to the smooth Markov perfect equilibrium developed by DeMarzo and He (2021). In our no commitment non-smooth equilibrium, equity holders gain positive tax benefits from leverage, there is a unique optimal leverage ratio as well as a unique optimal maturity of debt.

1 Introduction

Dynamic capital structure models recognize that the current price of a firm's debt depends on the future debt issuance policy of the firm. This interdependence between debt prices today and the firm's future debt issuance is further complicated by the fact that equity holders may not commit to a particular debt issuance policy. Future debt issuance not only affects how dynamic capital structure models value risky debt but also the tradeoffs firms face in recapitalizing over time.

The modeling of a firm's debt issuance policy varies across dynamic capital structure models. Some models fix the absolute level of a firm's debt and prohibit the firm from issuing additional debt. For example, Leland (1994) assumes that once issued, the face value of debt remains fixed. This assumption is justified by the fact that the issuance of additional debt is typically proscribed by bond covenants. Other models, for example, Fischer, Heinkel and Zechner (1989), Leland (1998), Goldstein, Ju and Leland (2001), Dangl and Zechner (2021), assume a firm maintains a constant level of debt until asset values rise sufficiently at which point the firm increases the face value of its debt outstanding but only by first repurchasing all existing debt subject to a call premium and issuance costs. In these models, firm debt level changes occur infrequently and are lumpy. This stands in contrast to DeMarzo and He (2021) who argue that in practice firms often borrow incrementally over time and posit a debt issuance policy in which the firm smoothly recapitalizes.

In addition to taxes and bankruptcy costs, issuance costs are an important determinant of a firm's restructuring decision especially when a firm's debt issuance policy calls for the repeated issuance of incremental debt. It is costly for firms to issue debt and these costs affect how frequently firms adjust their financing. While DeMarzo and He (2021) abstract from any issuance costs, Benzoni, Garlappi, Goldstein and Ying (2022) investigate a dynamic capital structure model with fixed issuance costs in which equity holders cannot commit to a future debt issuance policy. In their no commitment equilibrium, firms issue discrete amounts of debt at infrequent intervals and extract positive tax benefits doing so. Altinkiliç and Hansen (2000), however, provide evidence that fixed costs are not a large portion of the costs of underwriting debt issues, accounting for only approximately 10% of the underwriter spread. For the more relevant case of proportional debt issuance costs, Benzoni et al. (2022) claim that firms' recapitalization decisions are no longer lumpy but rather DeMarzo and He (2021)'s equilibrium obtains in which firms follow a smooth debt issuance policy and no tax benefits are extracted.

This paper introduces a dynamic capital structure model with proportional issuance costs in which a firm incrementally issues debt consistent with the leverage ratchet effect (Admati, DeMarzo, Helwig and Pfleiderer (2018)) whereby equity holders always have an incentive to borrow more but are never willing to voluntarily reduce leverage. Debt issuance in our model is increasing in a firm's interest coverage ratio, a widely used measure of financial soundness. This coverage ratio depends on the firm's operating cash flows (EBIT) as well as the firm's debt issuance and any debt buy backs, either voluntary or pursuant to a sinking fund provision. In the widely used framework that a firm's cash flows follow a lognormal process, we prove that it is optimal for the firm to issue debt when triggered by a sufficiently large increase in its interest coverage ratio, not to issue debt otherwise, and never to voluntarily buy back its debt.

Assuming equity holders commit to it, our debt issuance policy is consistent with the practice by which firms actually issue incremental debt. A firm's existing debt typically includes covenants prohibiting the firm from, among other actions, incurring additional indebtedness. However, circumstances can subsequently arise in which the firm needs to issue additional debt. Debt contracts recognize the costly nature of covenant renegotiation (Gamba and Mao (2020)) and consequently often include so-called baskets and carveouts (Ivashina and Vallee (2020)) that permit the firm access to additional debt but subject to specific conditions and negotiated limitations based on observable information, such as financial ratios, designed to protect the value of existing debt in the face of the firm's additional indebtedness. Commitment to the optimal debt issuance policy in our model corresponds to an agreement between equity holders and lenders to include a carveout to existing debt's covenant limiting additional debt which is triggered whenever the firm's interest coverage ratio reaches a contractually specified level given by the interest coverage ratio prevailing when the existing debt was originally issued. By way of example, given a firm's existing debt outstanding and ignoring any debt buy backs, achieving a new maximum EBIT increases the firm's interest coverage ratio beyond the level specified in the carveout at which point the firm in our model is incentivized to issue incremental debt which consequently reduces this ratio back to its trigger level.

By contrast, without commitment, a firm in our model may not be able to borrow at all because lenders do not have an incentive to lend to it. However, if the firm does borrow, we prove that it is also optimal for it to borrow incrementally only when triggered by a sufficiently large increase in its interest coverage ratio and never before. While, as expected, the firm borrows more without commitment, there is a limit to how much more the firm will borrow. Equity holders recognize that pursuing a more aggressive debt issuance policy may decrease the value of equity due to the resultant increased future debt burden exceeding the proceeds received from issuing the additional debt.

Our optimal debt issuance policy stands in contrast to that of DeMarzo and He (2021). Assuming an arbitrage relation between a firm's debt issues across varying levels of indebtedness, DeMarzo and He (2021) argue that the optimal issuance policy is continuous and restrict their attention to absolutely continuous (smooth) policies. In particular, they solve equity holders' Hamilton-Jacobi-Bellman (HJB) equation *assuming* equity holders find it optimal to adjust debt smoothly. We also follow a continuous policy in our no commitment model but our optimal policy is non-smooth - an initial jump at origination followed by continuous but singular issuances that occur at isolated points in time ("devil's staircase"). DeMarzo and He (2021) claim to rule out any non-smooth debt issuance in the case of log-normally distributed cash flows as in our model. However, this assertion relies on a *given* equilibrium issuance policy. If equity holders' proposed leverage change prompts creditors to alter their expectations of future issuance policy, creditors will not accept prices based on the previous issuance policy, thereby invalidating the assumed arbitrage relation and disproving their uniqueness claim.

Equity holders extract tax benefits from issuing debt in our dynamic capital structure model. Without issuance costs, a nonviable corner solution of 100% debt financing with infinitely short maturity debt is optimal. But issuance costs matter. With or without commitment, increasing proportional issuance costs, holding all other model parameters fixed, lengthens optimal maturity, lowers the optimal leverage ratio and widens the credit spread on issued debt. Under no commitment, a firm's optimal debt maturity is shorter, its optimal leverage ratio is higher and the credit spread on the firm's debt is wider than with commitment.

We apply our model to investigate why high yield indentures typically include covenants committing the issuer to limit the issuance of additional debt subject to carveouts of this restriction. By contrast, investment grade indentures do not prohibit additional debt issuance. To do so, we consider the counterfactual of an investment grade issuer deciding, alternatively, to commit to a debt issuance policy as well as the counterfactual of a high yield issuer deciding, alternatively, to not commit. We document that if high yield issuers did not commit, not only would their firm values fall, potentially precipitously, but creditors would demand far higher rates of interest or, worse still, would otherwise refuse to lend to them. While investment grade issuers could increase firm value by committing, by not doing so they avoid the costs of writing necessary indenture provisions and are also able to lever much more.

The plan of this paper is as follows. In Section 2, we review the limitations, if any, on a firm's ability to issue incremental debt. Investment grade firms typically do not face limitations while non-investment grade firms do. Our dynamic capital structure model is presented in Section 3. Assuming that the firm's EBIT is lognormal, we investigate optimal debt issuance and the pricing of debt both when equity holders do not commit to a debt issuance policy and when they do commit to a debt issuance policy maximizing firm value.

We prove that it is optimal for the firm to issue debt when triggered by a sufficiently large increase in its interest coverage ratio, not to issue debt otherwise, and never to voluntarily buy back its debt. We also compare our debt issuance policy to that of DeMarzo and He (2021). Section 4 investigates the properties of our dynamic capital structure model emphasizing the role of issuance costs in determining a firm's optimal capital structure and the optimal maturity of issued debt. A firm's decision of whether or not to commit to a predetermined debt issuance policy is explored in Section 5 and we conclude in Section 6.

2 Limitations on incurring indebtedness

The circumstances under which a firm can incur additional indebtedness are detailed in the covenants to its existing debt. Limitations on incurring indebtedness differ depending on whether the debt issue is investment grade or high yield.[1] High yield indentures often include covenants by which the issuer commits to limit the issuance of additional debt whereas investment grade indentures do not include such limitations. Under certain circumstances, however, high yield indentures do permit the firm to incrementally issue additional debt.

2.1 Investment grade debt

Investment grade bonds typically do not include covenants limiting a firm from incurring additional debt.[2] However, investment grade indentures often will limit the amount of secured debt an issuer can incur that would effectively be senior to the investment grade

[1] See, for example, PEI (2014), page 58.

[2] For example, Apple's Prospectus accompanying its 2013 issuance of $17 billion in notes states that "The indenture for the notes does not ... limit our ability to incur additional indebtedness that is secured, senior to or equal in right of payment to the notes." Page S-5.

bonds. The effectiveness of the negative pledge covenant in preventing the subsequent issuance of secured debt is, however, limited (see, for example, Bjerre (1999) and Donaldson, Gromb and Piacentino (2020)) because it can only be enforced against the issuer and not against the lender who takes a security interest in the firm's assets.

2.2 High yield debt

In contrast to investment grade indentures, high yield indentures typically include numerous covenants designed to preserve the issuer's ability to pay principal and interest on the bonds when due. Not only do these covenants limit incurring additional indebtedness, they also restrict, among other actions, the issuer's ability to grant liens on assets, make certain payments, such a dividends and stock repurchases, sell assets and enter into transactions with affiliates.

Ivashina and Vallee (2020) argue that the effectiveness of these provisions in high yield indentures[3] is frequently weakened by the inclusion of carveouts to particular restrictions as well as "baskets" which impose thresholds up until which a restriction does not hold. In particular, carveouts and baskets in high yield indentures typically permit the issuer access to additional debt but subject to specific conditions and negotiated limitations designed to protect existing lenders' claims. Issuers justify relaxing limitations to incurring additional indebtedness by the fact that they require the ability to incur additional debt in the ordinary course of business.[4] Without additional debt, a firm may not be able to pursue additional investment opportunities and the failure to do so could diminish the value of the existing debt.

As a result, high yield indentures typically permit issuers to subsequently increase indebt-

[3]Ivashina and Vallee (2020) focus on leveraged loan credit agreements. High yield indentures include similar contractual qualifications that weaken their various covenants.

[4]See, for example, PEI (2014), page 57.

edness by incurring ratio debt. That is, the issuer is prohibited from incurring additional debt unless a specific financial ratio, often a fixed charge coverage ratio, is met, typically at least a 2 to 1 ratio after accounting for the incurrence of the incremental debt.[5] Meeting the specified financial ratio is designed to ensure the issuer's ability to continue paying principal and interest on the high yield bonds in the face of the additional indebtedness. Ratio debt permitted by a high yield's indenture can be either *pari passu* or subordinate to the high yield issue.

If an issuer is unable to incur ratio debt, high yield indentures may also allow issuers to incur certain types and amounts of "permitted indebtedness". Among the various categories of permitted debt identified in a high yield bond's indenture, the largest and most important is the credit facilities basket that allows for debt under credit facilities to be incurred and secured, typically ranking senior to the bond itself.[6]

Lien covenants are also relevant to determining whether a high yield issuer has capacity to secure new debt. Like investment grade indentures, high yield indentures typically include a permitted liens basket that allow the issuer to pledge collateral to secure a limited amount of additional debt.[7]

[5] For example, the indenture, dated March 19 2021, of Mattel's 3.75% Senior Notes due 2029 states "The Issuer and any Restricted Subsidiary may incur Indebtedness . . . if the Fixed Charge Coverage Ratio for the Issuer and its Restricted Subsidiaries, calculated as of the date on which such additional Indebtedness is incurred . . . would have been 2.00 to 1.00 or greater." But "provided, further, that the aggregate amount of Indebtedness . . . that may be Incurred . . . pursuant to the foregoing . . . shall not exceed the greater of (x) $150 million and (y) 2.5% of Consolidated Total Assets, at any one time outstanding . . ." Section 3.3(a), page 48.

[6] For example, the indenture, dated June 9 2020, of Royal Caribbean's 9.125% Senior Notes due 2023 states that the Incurrence of Indebtedness covenant "shall not, however, prohibit the incurrence of . . . Indebtedness of the Issuer under the Credit Facilities (which, for purposes of this clause, shall exclude committed but undrawn amounts under the Existing ECA Facilities) in an aggregate principal amount at any time outstanding not to exceed $11,375 million." (page 56) where "the term Credit Facilities shall include any agreement or instrument . . . increasing the amount of Indebtedness incurred thereunder or available to be borrowed thereunder." (page 13)

[7] For example, the indenture of Mattel's 3.75% Senior Notes due 2029 permits "other Liens securing Indebtedness in an aggregate principal amount which, together with the aggregate outstanding principal amount of all other Indebtedness of the Issuer and its Subsidiaries secured by Liens permitted under the

3 The Model

3.1 Lognormal Cash Flows

We derive explicit solutions for a firm's debt and equity in the widely used framework of a lognormal process for the firm's cash flows. In particular, the firm's assets-in-place generate earnings before interest and taxes (EBIT) at a rate of Y which evolves exogenously according to geometric Brownian motion with drift μ and volatility σ:

$$dY_t = Y_t \mu dt + Y_t \sigma dZ_t. \tag{1}$$

Let $\mathcal{P}(Y, C)$ denote the endogenous value of *all* debt issued by the firm where C is the *total* coupon rate promised to the firm's bondholders while $\mathcal{V}(Y, C)$ denotes the endogenous value of the firm's equity. Like Leland (1994), we rely on the total coupon rate C to keep track of the firm's overall amount of debt outstanding. To do so, we normalize a bond to have a coupon rate of one per unit time. The firm's debt also contains an exponential sinking fund provision whereby the firm continuously buys back its debt at par, denoted by P, at a constant rate $\xi > 0$. As a result, the average maturity of the firm's debt is $1/\xi$.

The firm is organized at $t = 0$ when debt can initially be issued. Subsequently, the firm can issue incremental *pari passu* debt[8], determined by the control g, or, alternatively, voluntarily buy back outstanding debt, determined by the control h. Therefore, the dynamics

terms of this clause ... does not at the time such Liens are incurred exceed 10% of the Issuer's Consolidated Net Tangible Assets as shown on the most recent audited consolidated balance sheet of the Issuer and its Subsidiaries. (page 22)

[8] *Pari passu* incremental debt explicitly excludes the possibility of (i) contractual subordination—issuing incremental debt that is more senior than the bond itself; (ii) collateral subordination—issuing incremental debt that is secured by collateral and so would effectively rank senior to the bond itself; and (iii) temporal subordination—issuing incremental debt which matures or is otherwise payable prior to the bond and would also be effectively senior to the bond itself.

of the firm's total coupon rate C are given by

$$dC_t = (g_t - h_t - \xi C_t)dt. \tag{2}$$

Note that the controls for issuing debt, g, and for voluntarily buying back debt, h, are over and above any contractual buy back due to the existing debt's sinking fund provision. Incremental debt is issued at its prevailing market value, $\frac{\mathcal{P}(Y,C)}{C}$ per unit of coupon, while any debt voluntarily bought back by the firm is assumed to be at par, P.

Using standard no arbitrage arguments, the values of the firm's equity, $\mathcal{V}(Y, C)$, and debt, $\mathcal{P}(Y, C)$, can be shown to satisfy the following partial differential equations (PDEs):

$$\begin{aligned} r\mathcal{V}(Y,C) = \frac{1}{2}\sigma^2 Y^2 \mathcal{V}_{11}(Y,C) + \mu Y \mathcal{V}_1(Y,C) + (g - h - \xi C)\mathcal{V}_2(Y,C) \\ + (1-\tau_e)Y - (1-\tau_e)C - \xi PC + g(1-k)\frac{\mathcal{P}(Y,C)}{C} - hP \end{aligned} \tag{3}$$

and

$$\begin{aligned} r\mathcal{P}(Y,C) = \frac{1}{2}\sigma^2 Y^2 \mathcal{P}_{11}(Y,C) + \mu Y \mathcal{P}_1(Y,C) + (g - h - \xi C)\mathcal{P}_2(Y,C) \\ + (1-\tau_i)C + \xi PC - g\frac{\mathcal{P}(Y,C)}{C} + hP \end{aligned} \tag{4}$$

where r is the exogenously determined risk free rate of interest, τ_i is the income tax rate paid by debt holders on coupons received, τ_e is the effective tax rate on dividends paid to equity holders[9], and k is the proportional transaction cost levied on any debt issuance.

[9] $\tau_e = \tau_c + (1-\tau_c)\tau_d$ where τ_c is the corporate tax rate paid by the firm and τ_d is the tax rate paid by equity holders on dividends received. There is a tax advantage to debt as long as $\tau_e > \tau_i$.

3.1.1 No Commitment

To begin with, we assume that equity holders do *not* commit to a debt issuance policy. Equity holders' corresponding Hamilton-Jacobi-Bellman (HJB) equation is given by:

$$0 = \max_{g\geq 0, h\geq 0} \bigg\{ \frac{1}{2}\sigma^2 Y^2 \mathcal{V}_{11}(Y,C) + \mu Y \mathcal{V}_1(Y,C) - \xi C \mathcal{V}_2(Y,C) - r\mathcal{V}(Y,C) + (1-\tau_e)Y \\ - (1-\tau_e)C - \xi P C + g\Big((1-k)\frac{\mathcal{P}(Y,C)}{C} + \mathcal{V}_2(Y,C)\Big) - h\Big(P + \mathcal{V}_2(Y,C)\Big)\bigg\}.$$

When the firm is initially organized at $t = 0$, equity holders decide how much, if any, debt to issue. Given the linearity of the objective in the control g, the optimal debt issuance is bang-bang (see, for example, Davis and Norman (1990))[10]:

$$g = \begin{cases} \infty & \text{if } (1-k)\frac{\mathcal{P}(Y,C)}{C} + \mathcal{V}_2(Y,C) > 0 \\ 0 & \text{if } (1-k)\frac{\mathcal{P}(Y,C)}{C} + \mathcal{V}_2(Y,C) < 0. \end{cases} \tag{5}$$

Therefore, it is optimal for equity holders at origination to instantaneously issue debt so that the resultant total coupon rate, C, satisfies the first order condition

$$(1-k)\frac{\mathcal{P}(Y_0,C)}{C} + \mathcal{V}_2(Y_0,C) = 0. \tag{6}$$

That is, equity holders issue debt infinitely fast so long as the marginal proceeds received from issuing additional debt net of issuance costs, $(1-k)\frac{\mathcal{P}(Y_0,C)}{C}$, exceeds the marginal loss in equity value from their obligation to service this additional debt, $-\mathcal{V}_2(Y_0,C)$.

[10] Relying on a no arbitrage argument, DeMarzo and He (2021) rule out a bang-bang debt issuance policy in the case of lognormally distributed firm cash flows. Their argument, however, assumes that the equilibrium debt price does not react to a change in the debt issuance policy. But, as we show later, if the leverage change alters creditors' beliefs about future debt issuance, creditors will update out-of-equilibrium debt prices and invalidate DeMarzo and He's no arbitrage relation and consequently their uniqueness proof.

Equity holders do not, in general, have an insatiable appetite to issue debt under a no commitment policy. It is *not* optimal to issue additional *pari passu* debt if the loss in equity value resulting from the obligation to service this additional debt exceeds the additional proceeds received.

An internal solution, C^*, to the first order condition, equation (6), exists for most realistic parameter values and our subsequent analysis will concentrate on this typical case. However, there are parameter values for which the first order condition is not satisfied. In this case, equity holders not committing to a debt issuance policy may not be able to issue debt at all. See Appendix A for more details on solving equation (6) and whether a firm can issue debt without commitment in our model.

Because the firm's cash flows are lognormally distributed and all model parameters are proportional to the cash flows, we conjecture that the value functions $\mathcal{P}(Y, C)$ and $\mathcal{V}(Y, C)$ are homogenous of degree one in Y and C and the internal solution to the first order condition, equation (6), is linear in Y_0. That is, there exists an optimal debt issuance parameter γ^*, independent of Y_0, such that $C_0^* = \gamma^* Y_0$. The firm's interest coverage ratio when debt is initially issued is therefore given by $\frac{1}{\gamma^*}$.

At a later date, $t > 0$, the firm's coupon rate absent recapitalization may be strictly less than $\gamma^* Y_t$ or, equivalently, the firm's interest coverage ratio may exceed $\frac{1}{\gamma^*}$. This can occur because the firm's cash flow rate Y_t has increased or because the firm's coupon rate has decreased due, for example, to the contractual repayment of maturing debt.

Since the model is time homogeneous with only proportional issuance costs and any incremental debt issued is *pari passu*, equity holders will now optimally issue incremental debt so that the resultant total coupon rate of all outstanding debt C_t^* equals $\gamma^* Y_t$. Doing so, the firm's interest coverage ratio reverts back to $\frac{1}{\gamma^*}$.

It should be noted that debt issued by the firm in our model, initially or incrementally, is always at the same price per unit of coupon. That is, since the total coupon rate is $\gamma^* Y_t$, for all those dates, $t \geq 0$, when the firm issues debt, debt is always issued at the price

$$\frac{\mathcal{P}(Y_t, \gamma^* Y_t)}{\gamma^* Y_t} = \frac{\mathcal{P}(1, \gamma^*)}{\gamma^*}$$

per unit of coupon. Hence, we will define this price to be the par value, P, of debt:

$$P = \frac{\mathcal{P}(1, \gamma^*)}{\gamma^*}. \tag{7}$$

With this definition, all debt in our model is indeed issued at par and, consistent with the U.S. tax code, there are no adjustments or limits to the full tax deductibility of coupon payments at the firm level.

Having investigated the case when $\gamma^* Y_t$ exceeds C_t, we turn our attention to the case when C_t may be strictly greater than $\gamma^* Y_t$ or, equivalently, the firm's interest coverage ratio may be less than $\frac{1}{\gamma^*}$ because the firm's cash flow rate Y_t has decreased. In this case, equity holders will not issue incremental debt. Rather, they will either simply let the firm's existing debt mature according to its sinking fund schedule or, otherwise, buy back the firm's debt voluntarily using the control h. However, we now demonstrate that it is not optimal for equity holders to buy back the firm's debt voluntarily. That is, equity holders find it optimal to set the control h equal to zero at all times.

From equity holders' HJB equation, the optimal buy back policy without commitment is also bang-bang:

$$h = \begin{cases} \infty & \text{if } P + \mathcal{V}_2(Y, C) < 0 \\ 0 & \text{if } P + \mathcal{V}_2(Y, C) > 0. \end{cases} \tag{8}$$

We will argue, using proof by contradiction, that $P + \mathcal{V}_2(Y, C)$ is never strictly negative and, hence, the control h should be set equal to zero at all times.

Assuming a Y and a C such that $P + \mathcal{V}_2(Y, C) < 0$, it would be optimal for equity holders when the firm's EBIT hits this Y to instantaneously buy back debt infinitely fast so that the resultant total coupon rate, C, satisfies the first order condition

$$P + \mathcal{V}_2(Y, C) = 0.$$

That is, equity holders would buy back debt infinitely fast so long as the marginal increase in equity value per unit decrease in coupon owed by the firm, $-\mathcal{V}_2(Y, C)$, exceeds P, the cost the firm incurs in buying back the debt at par value. This leads to a trigger policy similar to that just explored when the firm issues additional debt. In other words, there would exist an optimal debt buy back parameter, f, such that whenever C exceeds fY, the firm would buy back debt at par value until $C = fY$. That is, debt buy back would be triggered when the firm's cash flow rate, Y, falls to a lower boundary $\frac{C}{f}$.

Consequently, as Y falls, the firm would buy back all of its debt before Y ever hits the even lower strategic default boundary, implying that the firm never defaults. By holding out, therefore, creditors would always get their promised coupons under the assumed debt buy back policy. Because equity holders must offer creditors who accept to be bought out the same price as creditors who decide to hold out, equity holders can only buy back debt by offering the risk free value of the promised coupons. Hence, the par value must be equal to the risk free value of debt:

$$P = \frac{1 - \tau_i}{r}.$$

By contrast, because the debt is risk free, the marginal benefit to equity holders of reducing its coupon obligation by one unit would only be $\frac{1-\tau_e}{r}$ and so it will always be the case that

$P + \mathcal{V}_2(Y, C) > 0$, hence contradicting that the first order condition held and proving by contradiction that equity holders would never voluntarily buy back debt and that $h = 0$ at all times.

Since equity holders never find it optimal to voluntarily buy back debt under no commitment, the strategic default of the firm will be triggered if the firm's cash flows fall sufficiently. In particular, when the firm's cash flows fall below a lower trigger, $Y_b(C)$, equity holders will stop servicing the debt and the firm will default. Once again, this default trigger is conjectured to be linear in C because the firm's cash flows are lognormal and all parameters are proportional to cash flows. As a result, there exists a bankruptcy parameter, b, independent of C such that $Y_b(C) = \frac{C}{b}$.

Figure 1 provides a schematic representation of our model. For simplicity, we ignore the effect of debt maturing due to the sinking fund provision. Given an initial EBIT of Y_0, the firm at its origination issues debt so that $C_0^* = \gamma^* Y_0$. The firm does not recapitalize if its EBIT subsequently does not exceed Y_0. However, if the firm's EBIT falls sufficiently and hits the bankruptcy boundary $Y_{b_0} = \frac{C_0}{b}$ then the firm immediately defaults. Alternatively, the firm issues incremental debt if its EBIT subsequently evolves to exceed Y_0. Here Y_t is the firm's subsequent maximum EBIT and the firm issues incremental debt at each new maximum EBIT realized along EBIT's sample path from Y_0 to Y_t, following the green arrow along the line with slope γ^*, so that, as a result, the firm's total coupon accumulates to $C_t^* = \gamma^* Y_t$. Notice that with the now larger debt load, the firm's default is triggered at a correspondingly higher bankruptcy boundary, $Y_{b_t} = \frac{C_t}{b}$.

To keep track of the firm's total outstanding debt over time, we introduce the variable

$$M_t = \max_{s \leq t} Y_s e^{-\xi(t-s)} \tag{9}$$

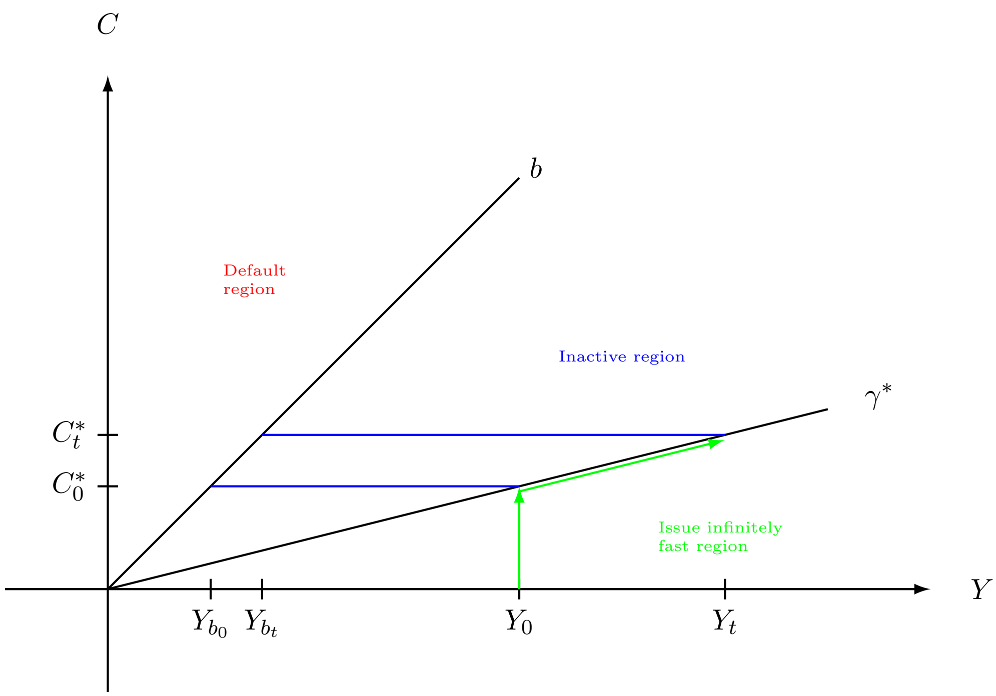


Figure 1: Graphical illustration of the model. Here γ^*, the optimal debt issuance parameter, is equal to the slope of the line separating the issue infinitely fast region from the inactive region, and b, the bankruptcy parameter, is equal to the slope of the line separating the inactive region from the default region.

which represents the firm's maximum cash flow rate through time $t \geq 0$, but adjusted for its existing debt's sinking fund provision.

The previous discussion gives that under no commitment the firm follows a debt issuance policy that ensures the firm's total coupon rate satisfies

$$C_t^* = \gamma^* M_t \tag{10}$$

for all $t \geq 0$.[11]

New debt is incrementally issued only when M increases. As we show later, M is characterized by a continuous sample path but the set of time points where M increases has measure zero.[12] Therefore, our debt issuance policy without commitment follows a singular process: a jump in the firm's leverage at its origination followed by the continuous but singular ("devil's staircase") issuance of incremental debt when M increases at distinct time points of measure zero.

3.1.2 Commitment

Equity holders can, alternatively, commit to the firm's future debt issuance choices. Doing so, equity holders agree to a debt issuance policy maximizing the value of the firm, $\mathcal{V}(Y, C) + \mathcal{P}(Y, C)$, as opposed to maximizing the value of their equity.

[11] We have now described the debt issuance policy when the model is originated at date zero, $t = 0$, where the EBIT is Y_0 for a firm that has no coupon obligations at origination. For completeness, we also specify the debt issuance policy, if we originate the model at an arbitrary date, t_0, where the EBIT is Y_{t_0} and the firm's total coupon rate is (already) C_{t_0} (due to past debt issuance). Define M_{t_0} to be $M_{t_0} = \frac{C_{t_0}}{\gamma^*}$ at this arbitrary date, t_0. If now $Y_{t_0} > M_{t_0}$, then the firm should issue additional debt infinitely fast until the total coupon rate is $C_{t_0} = \gamma^* Y_{t_0}$ and then M_{t_0} should be redefined to be $M_{t_0} = Y_{t_0}$. On the other hand, if $Y_{t_0} < \frac{C_{t_0}}{b} = \frac{\gamma^*}{b} M_{t_0}$, then the firm should default. Finally, if $\frac{C_{t_0}}{b} = \frac{\gamma^*}{b} M_{t_0} \leq Y_{t_0} \leq \frac{C_{t_0}}{\gamma^*} = M_{t_0}$, then the firm should stay passive, i.e., it should neither issue additional debt nor default, but just let time pass.

[12] See Section 3.2 below.

From the corresponding HJB equation[13],

$$
\begin{aligned}
0 = \max_{g\geq 0, h\geq 0} \Big\{ \frac{1}{2}\sigma^2 Y^2 \Big(\mathcal{V}_{11}(Y,C) + \mathcal{P}_{11}(Y,C)\Big) + \mu Y \Big(\mathcal{V}_1(Y,C) + \mathcal{P}_1(Y,C)\Big) \\
- \xi C \Big(\mathcal{V}_2(Y,C) + \mathcal{P}_2(Y,C)\Big) - r\Big(\mathcal{V}(Y,C) + \mathcal{P}(Y,C)\Big) \\
+ (1-\tau_e)Y + (\tau_e - \tau_i)C + g\Big(\mathcal{V}_2(Y,C) + \mathcal{P}_2(Y,C) - k\frac{\mathcal{P}(Y,C)}{C}\Big) \\
- h\Big(\mathcal{V}_2(Y,C) + \mathcal{P}_2(Y,C)\Big)\Big\},
\end{aligned}
$$

the optimal debt issuance policy with commitment is once again bang-bang:

$$
g = \begin{cases} \infty & \text{if } \mathcal{P}_2(Y,C) + \mathcal{V}_2(Y,C) > k\frac{\mathcal{P}(Y,C)}{C} \\ 0 & \text{if } \mathcal{P}_2(Y,C) + \mathcal{V}_2(Y,C) < k\frac{\mathcal{P}(Y,C)}{C}. \end{cases} \tag{11}
$$

Now it is optimal for equity holders at origination to instantaneously issue debt so that the resultant total coupon rate, C, satisfies the first order condition

$$
\mathcal{P}_2(Y_0,C) + \mathcal{V}_2(Y_0,C) = k\frac{\mathcal{P}(Y_0,C)}{C}. \tag{12}
$$

Satisfying this first order condition in C gives the first best debt issuance policy that maximizes total firm value.

Similar to the no commitment case, with commitment there exists an optimal debt issuance parameter, denoted by γ^{**}, independent of Y_0 such that $C_0^{**} = \gamma^{**}Y_0$. The firm's interest coverage ratio when debt is initially issued with commitment is given by $\frac{1}{\gamma^{**}}$. Furthermore, whenever $C_t < \gamma^{**}Y_t$, the firm commits to issue incremental *pari passu* debt instantaneously up to $C_t^{**} = \gamma^{**}M_t$.[14]

[13]Obtained by simply adding the HJB equations for equity and for debt.

[14]Footnote 11 also applies to explain how to complete the debt issuance policy with commitment if we

Since $\mathcal{P}(Y, C)$ is a concave function in C and $\mathcal{P}(Y, 0) = 0$, it follows that $\mathcal{P}_2(Y, C) < \frac{\mathcal{P}(Y,C)}{C}$ and hence $\gamma^{**} < \gamma^*$. That is, equity holders are expected to borrow less aggressively with commitment or, equivalently, the firm will borrow less and at a higher interest coverage ratio with commitment than without commitment.

This debt issuance policy with commitment can immediately be seen to correspond to the issuance of ratio debt with a contractually specified interest coverage ratio[15] of $\frac{1}{\gamma^{**}}$. In other words, equity holders and lenders agree to a carveout of the covenant limiting additional debt triggered at an interest coverage ratio of $\frac{1}{\gamma^{**}}$. For example, in the absence of a sinking fund provision, when a new maximum EBIT is reached given a firm's existing debt outstanding, the firm's interest coverage ratio increases beyond $\frac{1}{\gamma^{**}}$, and the firm incrementally issues *pari passu* debt so that after its issuance, the firm's interest coverage ratio is consequently reduced to the contractually specified trigger value of $\frac{1}{\gamma^{**}}$.

As in the no commitment case, bankruptcy will also be triggered when the firm's cash flow rate hits a lower boundary $Y_b(C)$, now lower than in the no commitment case as the issuance policy is less aggressive.

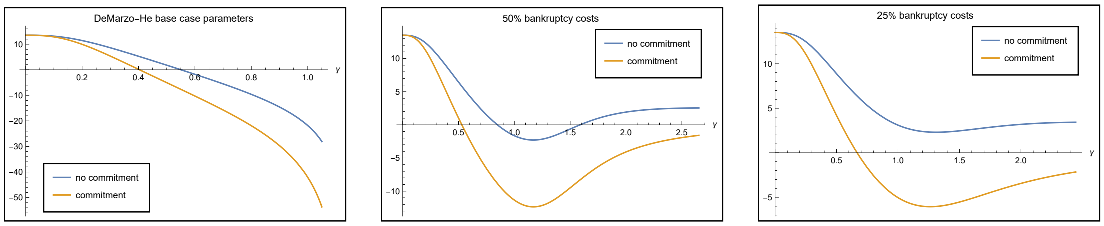
3.1.3 Illustrative Examples

To illustrate properties of our model, we choose parameter values used by DeMarzo and He (2021) in the case where the firm's cash flows follow a lognormal process: $r = 5\%$, $\mu = 2\%$, $\sigma = 40\%$, $\tau_i = 0\%$, $\tau_e = 30\%$, $\xi = .2$ (corresponding to 5-year average maturity debt), and finally, we also assume, like DeMarzo and He (2021), no issuance costs, $k = 0$, as well as a zero recovery value in bankruptcy, $\alpha = 100\%$.

For these parameter values, Panel (a) of Figure 2 plots the first order conditions for opti-

originate the model at an arbitrary date t_0.

[15] In our model, the interest coverage ratio is equivalent to a fixed charge coverage ratio as interest payments are the only fixed charge payable by the firm.



(a) Base case (b) $\alpha = .50$ (c) $\alpha = .25$

Figure 2: First order condition.

mal debt issuance, with commitment (yellow line) versus without commitment (blue line), against potential γ values, equations (12) and (6), respectively. We see that our model's optimal debt issuance parameter without commitment, $\gamma^* = 0.551$, exceeds its optimal debt issuance parameter with commitment, $\gamma^{**} = 0.404$. While equity holders borrow more without commitment, they clearly do not have an incentive to follow a debt issuance policy characterized by any issuance parameter greater than γ^* because the decrease in the value of equity due to the resultant increased future debt burden will exceed the proceeds received from issuing the additional debt.

To further investigate properties of our model, bankruptcy costs are reduced to $\alpha = 50\%$ in Panel (b) of Figure 2 and further reduced to $\alpha = 25\%$ in Panel (c) of Figure 2, while holding all other parameter values the same. Intuitively, reducing bankruptcy costs results in higher recovery values for debt in bankruptcy and, all else being equal, increases the current value of debt and lowers the credit spread at which debt holders are willing to lend. The marginal proceeds of issuing another dollar of coupon are now higher, incentivizing equity holders to pursue a more aggressive debt issuance policy.

The incentive to issue more debt given lower bankruptcy costs in the case where the firm commits to its future debt issuance policy is evident in Figure 2. As compared to an optimal debt issuance parameter with commitment of $\gamma^{**} = 0.404$ for $\alpha = 100\%$ in Panel (a), the optimal parameter increases to $\gamma^{**} = 0.530$ for $\alpha = 50\%$ in Panel (b), and increases even further to $\gamma^{**} = 0.667$ for $\alpha = 25\%$ in Panel (c).

By contrast, if equity holders do not commit to a future debt issuance policy, thereby ignoring how their actions affect existing creditors, lenders understand that the lower bankruptcy costs may trigger an insatiable appetite by equity holders to issue additional debt and prompt lenders not to lend to the firm at all. Equivalently, lenders in this case would only be willing to buy additional debt from the firm at a price of zero and so *de facto* preclude equity holders from issuing additional debt.

To see this, consider Panel B of Figure 2 where for lower bankruptcy costs, α=50%, we now have two solutions of equity holders' first order condition for optimal debt issuance without commitment as the corresponding first order condition crosses the horizontal axis twice. As we argue below, the debt issuance policy corresponding to the first crossing point, γ^*=0.848, represents the only Markov equilibrium issuance policy as lenders believe that equity holders are incentivized to adhere to this policy. If the firm attempts to issue debt too aggressively, consistent with the second crossing point, lenders believe that equity holders here have an incentive not to adhere to this policy but to continue to issue debt until bankruptcy and so respond by not lending.

To understand why these are rational responses by lenders, assume the firm chooses a particular debt issuance policy indexed by a value of γ. If lenders believe that equity holders will adhere to this policy, they will price the firm's debt according to this γ value. In equilibrium, equity holders have no incentive to deviate from this issuance policy nor do lenders have an incentive to change the price they are willing to offer for the firm's debt, taking as given the other's decision.[16,17]

In the case of Panel B of Figure 2, lenders understand that for all γ values to the left of the first crossing point, equity holders will increase the aggressiveness with which they issue debt until the issuance policy $\gamma^* = 0.848$ is reached. Once here, lenders recognize that equity holders do not have an incentive to deviate from this issuance policy and so are willing to lend to the firm at the debt price corresponding to γ^*. For example, if Y were now to increase, effectively reducing the firm's debt issuance parameter below γ^*, lenders recognize that equity holders are incentivized to issue more debt but not alter their issuance policy until their first order condition is re-established at a total coupon rate of $\gamma^* M$. Alternatively, if Y were to decrease, as we have shown earlier, lenders understand that in this case equity holders have no incentive to buy back their debt and so will also not alter their issuance policy.

By contrast, the debt issuance policy corresponding to the second crossing point in Panel B of Figure 2 is not a Markov equilibrium issuance policy. Lenders recognize that equity holders have an incentive here to deviate from this policy and so will not lend to the firm at the debt price corresponding to this debt issuance parameter. In fact, they will not lend additional debt to the firm at all. To see this, assume Y were now to decrease, effectively increasing the firm's debt issuance parameter above the debt issuance parameter corresponding to the second crossing point. Given the positive first order condition here, equity

[16] Figure 2 simply plots equity holders' first order conditions, with commitment and without commitment, as a function of γ for the assumed model parameters. It does not take into account whether any of the two parties have incentives to deviate from the proposed debt issuance policy. That is, whether equity holders would deviate by choosing a different issuance strategy than given by the assumed γ or whether lenders would deviate by setting a different price at which they would accept to buy the debt the firm is issuing than the price corresponding to the policy γ.

[17] We extend our debt and equity value functions in Appendix B to accommodate the possibility of out-of-equilibrium debt issuance by a firm's equity holders. We also confirm that our model's debt issuance and debt buy back policies represent Markov perfect equilibrium policies as equity holders prefer not to issue incremental debt in the continuation region nor do they have an incentive to buy back their debt.

holders now have an incentive to increase the aggressiveness with which they issue debt. Furthermore, there is no upper bound to their incentive to increase issuance aggressiveness. Lenders understand that this behavior by equity holders will lead to the firm's bankruptcy and respond by not lending to the firm, or equivalently, only willing to buy newly issued debt at a price of zero. At a debt price of zero, equity holders' first order condition is no longer positive and, hence, they will abandon this aggressive issuance policy.

This consequent breakdown in the debt market means the firm's only feasible debt issuance policy without commitment corresponds to the lower debt issuance parameter, γ^*=0.848, which, once again, exceeds the optimal debt issuance parameter with commitment γ^{**}=0.530.

Bankruptcy costs are further reduced to α=25% in Panel C of Figure 2. While the firm can still borrow with commitment (γ^{**}=0.667), it no longer can borrow without commitment. Lenders realize that in this case equity holders' first order condition remains positive for all potential issuance policies γ. As in the prior case, lenders respond to equity holders' unbounded incentive to increase debt issuance by refusing to lend to the firm or, equivalently, only buy newly issued debt at a price of zero. In the face of this market breakdown, the firm will remain all equity financed, consistent with the zero-leverage puzzle of Strebulaev and Yang (2013).

3.2 Model Solution

It will be convenient to work with cash flows scaled by M_t,

$$y_t \equiv Y_t/M_t \in (0, 1]. \tag{13}$$

Note that unlike DeMarzo and He (2021) or Benzoni et al. (2022), we do not scale cash flows by an endogenous control variable.[18]

Within our setting, debt and equity values are homogenous of degree one so that

$$\mathcal{P}(Y_t, C_t) = M_t \mathcal{P}(\frac{Y_t}{M_t}, \frac{C_t}{M_t}) = M_t p(y_t, \gamma) \tag{14a}$$

and

$$\mathcal{V}(Y_t, C_t) = M_t \mathcal{V}(\frac{Y_t}{M_t}, \frac{C_t}{M_t}) = M_t v(y_t, \gamma). \tag{14b}$$

We provide explicit solutions for the scaled debt value $p(y_t, \gamma)$ and scaled equity value $v(y_t, \gamma)$. To do so, requires we specify the evolution of the scaled cash flow rate y when the firm's cash flow rate Y evolves according to equation (1).

From Shepp and Shiryaev (1995),[19] in general, the maximum of a geometric Brownian motion is a nondecreasing process of locally bounded variation that only increases with probability zero.[20] Specializing their results to our case, both M and y have continuous sample paths, y is a Markov process, and y and M are diffusions that can be specified as

$$dM_t = -M_t \xi dt \tag{15}$$

$$dy_t = y_t(\mu + \xi)dt + y_t \sigma dZ_t \tag{16}$$

with $y_t = 1$ being an instant reflection point for y.

As equation (16) shows, scaled cash flows have the same volatility as the cash flows them-

[18] DeMarzo and He (2021) and Benzoni et al. (2022) scale EBIT by total issued principal F.

[19] See Shepp and Shiryaev (1995), in particular, pages 109-111. Shepp and Shiryaev (1995) are concerned with the pricing of a "Russian" option where the underlying asset follows a lognormal distribution.

[20] More precisely, $\int_0^t \mathbb{1}_{\{y_s=1\}} ds = 0$ almost surely for any t.

selves, Y. This follows because M is locally deterministic. The drift of the scaled cash flow rate is increased by the rate of debt amortization ξ.[21]

In our model, when the scaled cash flow rate reaches the upper boundary, y_t=1, the firm issues incremental *pari passu* debt instantaneously until C_t=γM_t, γ=$\{\gamma^*, \gamma^{**}\}$. As a result, C_t=γM_t at all times meaning that observing the total coupon rate C_t reveals $M_t = \frac{C_t}{\gamma}$.

Using equation (14b), we can derive the following partial derivatives:

$$\mathcal{V}_1(Y, C) = \frac{d}{dY}\frac{C}{\gamma}v(\frac{\gamma Y}{C}, \gamma) = v'(\frac{\gamma Y}{C}, \gamma) = v'(y, \gamma) \tag{17}$$

$$\mathcal{V}_{11}(Y, C) = \frac{d}{dY}v'(\frac{\gamma Y}{C}, \gamma) = \frac{\gamma}{C}v''(y, \gamma) \tag{18}$$

$$\mathcal{V}_2(Y, C) = \frac{d}{dC}\frac{C}{\gamma}v(\frac{\gamma Y}{C}, \gamma) = \frac{1}{\gamma}v(y, \gamma) - \frac{Y}{C}v'(y, \gamma) = \frac{1}{\gamma}v(y, \gamma) - \frac{y}{\gamma}v'(y, \gamma). \tag{19}$$

Corresponding results hold for $\mathcal{P}$ when using equation (14a). Substituting these results into equation (3) and multiplying through by $\frac{\gamma}{C}$ gives that the scaled equity value v for a given γ satisfies the following ordinary differential equation:

$$(r{+}\xi)v(y, \gamma) = \frac{1}{2}\sigma^2 y^2 v''(y, \gamma) + (\mu{+}\xi)yv'(y, \gamma) + (1{-}\tau_e)y - (1{-}\tau_e)\gamma - \xi P\gamma \tag{20}$$

where, as before, r is the exogenously determined risk free rate of interest, τ_e denotes the effective tax rate on dividends paid to equityholders and P is the principal balance of a bond issued by the firm at par with a coupon of \$1 per unit time. Similarly, the scaled

[21] Scaled cash flows in our model evolve as in the DeMarzo and He (2021) model with lognormal cash flows except for the fact their drift is also decreased by the rate of debt issuance. However, unlike DeMarzo and He (2021), we only issue debt at the boundary y_t=1 and so do not require this adjustment.

debt value p satisfies the following ordinary differential equation

$$(r{+}\xi)p(y,\gamma) = \frac{1}{2}\sigma^2 y^2 p''(y,\gamma) + (\mu{+}\xi)yp'(y,\gamma) + (1{-}\tau_i)\gamma + \xi P\gamma \tag{21}$$

where τ_i denotes the income tax rate paid by debt holders on coupons received. Note that both controls g and h are zero in the continuation region where these ordinary differential equations apply as the firm only issues debt at the boundary $y_t = 1$, never issues debt when $y_t < 1$, and never buys back its debt.

When the scaled cash flow rate y_t falls below the endogenous default boundary $y_b{\in}(0,1)$, equity holders are no longer willing to service the outstanding debt and choose to strategically default. Debt holders take over the firm net of bankruptcy costs leading to the following lower boundary conditions for debt and equity:

$$v(y_b,\gamma) = 0 \tag{22a}$$

$$v'(y_b,\gamma) = 0 \tag{22b}$$

and

$$p(y_b,\gamma) = (1-\alpha)\frac{(1-\tau_e)y_b}{r-\mu}. \tag{22c}$$

Equation (22a) is the equity holders' value matching condition that equity is worthless at default, while equation (22b) gives the corresponding smooth pasting condition. Equation (22c) is the debt holders' value matching condition that debt holders receive the going concern value of the firm $\frac{(1-\tau_e)y_b}{r-\mu}$, net of bankruptcy costs $\alpha > 0$, at the default-triggering level y_b.

To determine the boundary conditions for debt and equity at the upper boundary $y_t{=}1$, we consider the case in which the firm issues additional *pari passu* debt when y_t=1+Δ and let $\Delta \to 0$. Given proportional issuance costs $k>0$, we have the following boundary conditions at $y = 1 + \Delta$:

$$v(1+\Delta,\gamma) = \underbrace{(1+\Delta)v(1,\gamma)}_{\text{scaled equity}} + (1-k) \times \underbrace{\frac{\Delta}{1+\Delta}}_{\text{fraction newly issued debt}} \times \underbrace{(1+\Delta)p(1,\gamma)}_{\text{scaled total debt}}$$

$$p(1+\Delta,\gamma) = \underbrace{\frac{1}{1+\Delta}}_{\text{fraction old debt}} \times \underbrace{(1+\Delta)p(1,\gamma)}_{\text{scaled total debt}}.$$

That is, equity and total debt values are scaled up as the firm is issuing new *pari passu* debt. The fraction of the total debt that is newly issued is sold at its market price and the proceeds accrue to equity holders net of issuance costs. Existing debt holders hold only the remaining fraction, $\frac{1}{1+\Delta}$, of the total debt after the new debt is issued. Re-arranging these expressions and dividing by Δ gives

$$\begin{aligned} \frac{v(1+\Delta,\gamma) - v(1,\gamma)}{\Delta} &= v(1,\gamma) + (1-k)p(1,\gamma) \\ \frac{p(1+\Delta,\gamma) - p(1,\gamma)}{\Delta} &= 0. \end{aligned}$$

Letting $\Delta \to 0$, the upper boundary conditions for debt and equity simplify to

$$v'(1,\gamma) = v(1,\gamma) + (1-k)p(1,\gamma) \tag{23a}$$

$$p'(1,\gamma) = 0. \tag{23b}$$

Because of scale invariance, equity holders' value matching condition, equation (23a), gives that the marginal increment in the value of firm at the upper boundary totally accrues to equity holders including the debt proceeds net of issuance costs. Existing bond holders do not benefit, equation (23b), since the issued debt is *pari passu*.

Proposition

For a given debt policy parameter γ and a given maturity structure parameter ξ, the debt and equity value functions can be solved as

$$p(y,\gamma) = \frac{(1-\tau_i+\xi P)\gamma}{r+\xi} + \left((1-\alpha)\frac{1-\tau_e}{r-\mu}y_b - \frac{(1-\tau_i+\xi P)\gamma}{r+\xi}\right)\frac{x_1 y^{x_2} - x_2 y^{x_1}}{x_1 y_b^{x_2} - x_2 y_b^{x_1}} \quad (24)$$

$$\begin{aligned} v(y,\gamma) = & \frac{1-\tau_e}{r-\mu}y - \frac{1-\tau_e}{r-\mu}y_b \frac{(x_1-1)y^{x_2}-(x_2-1)y^{x_1}}{(x_1-1)y_b^{x_2}-(x_2-1)y_b^{x_1}} \\ & - \frac{(1-\tau_e+\xi P)\gamma}{r+\xi}\left(1+\frac{(x_2-1+y_b^{x_2})y^{x_1}-(x_1-1+y_b^{x_1})y^{x_2}}{(x_1-1)y_b^{x_2}-(x_2-1)y_b^{x_1}}\right) \\ & + (1-k)P\gamma \frac{y_b^{x_2}y^{x_1}-y_b^{x_1}y^{x_2}}{(x_1-1)y_b^{x_2}-(x_2-1)y_b^{x_1}} \end{aligned} \quad (25)$$

where $x_1 = \frac{(\frac{1}{2}\sigma^2-\mu-\xi)+\sqrt{(\mu+\xi-\frac{1}{2}\sigma^2)^2+2(r+\xi)\sigma^2}}{\sigma^2} > 1$, $x_2 = \frac{(\frac{1}{2}\sigma^2-\mu-\xi)-\sqrt{(\mu+\xi-\frac{1}{2}\sigma^2)^2+2(r+\xi)\sigma^2}}{\sigma^2} < 0$ and the default boundary $y_b(\gamma,\xi)$ must be numerically solved for. □

These expressions value the firm's total debt and equity explicitly taking into account that the firm issues incremental debt as its earnings increase sufficiently but subsequently defaults if its earnings decrease sufficiently.

To better understand the debt value function, rewrite it as

$$p(y,\gamma) = \underbrace{\frac{(1-\tau_i+\xi P)\gamma}{r+\xi}}_{\text{Value of promised payments}} - \underbrace{\left(\frac{(1-\tau_i+\xi P)\gamma}{r+\xi} - (1-\alpha)\frac{1-\tau_e}{(r-\mu)}y_b\right)\mathcal{C}_1(y,y_b)}_{\text{Loss due to default}}$$

where $\mathcal{C}_1(y, y_b)$ is the value given scaled earnings y of a claim paying \$1 when scaled earnings hit y_b. This claim takes into account the fact that if scaled earnings hit the upper boundary y=1 before hitting y_b, then y_b is correspondingly adjusted upwards due to instant reflection (see Figure 1).

In the case of the equity value function, we have

$$v(y,\gamma) = \underbrace{\frac{1-\tau_e}{r-\mu}y}_{\text{Unlevered firm value}} - \underbrace{\frac{1-\tau_e}{r-\mu}y_b\mathcal{C}_2(y,y_b)}_{\text{Loss of unlevered firm value due to default}} - \underbrace{(1-\tau_e+\xi P)\gamma\mathcal{C}_3(y,y_b)}_{\text{Current and future payments to debt holders}} + \underbrace{(1-k)P\gamma\mathcal{C}_4(y,y_b)}_{\text{Proceeds from future debt issues}} .$$

Here $\mathcal{C}_2(y, y_b)$ is the value given scaled earnings y of a claim paying $\$1 \times \frac{M_\tau}{M_0}$ when scaled earnings hit y_b where $\frac{M_\tau}{M_0}$ is the fractional increase in earnings between the time when the claim was originated and when scaled earnings hit y_b. The final payment is always \$1 for $\mathcal{C}_1$ when scaled earnings hit y_b, whereas for $\mathcal{C}_2$ the final payment is adjusted upwards every time scaled earnings hit the upper boundary y=1 before hitting y_b. In particular, if M=M_τ when scaled earnings hit y_b but M_0<M_τ when the claim was originated, then the contingent payoff is given by $\$1 \times \frac{M_\tau}{M_0}$. This is not the case for $\mathcal{C}_1$ where the payment is always \$1 when scaled earnings hit y_b.

Similarly, $\mathcal{C}_3(y, y_b)$ is the value given scaled earnings y of a claim paying a flow of payments at the rate $\$1 \times \frac{M_t}{M_0}$ per unit of time until the scaled earnings hit y_b. The payment rate of this claim is similar to that of $\mathcal{C}_2$ in that it is adjusted upwards every time scaled earnings hit the upper boundary y=1 before hitting y_b. Moreover, this payment rate is adjusted downwards over time according to the debt's assumed sinking fund provision. This claim, therefore, reflects the coupon and sinking fund obligations of the firm, both its original debt as well as any incremental debt issued in the future.

Finally, $\mathcal{C}_4(y, y_b)$ is the value given scaled earnings y of a claim making payments of $\$1 \times \frac{dM_t}{M_0}$ each time scaled earnings hit the upper boundary $y{=}1$ before hitting y_b and reflects the incremental proceeds to the firm of issuing any incremental debt in the future.

3.3 Comparison to the Debt Issuance Policy of DeMarzo and He (2021)

Motivated by the observation that firms often borrow incrementally, DeMarzo and He (2021) do not allow for bang-bang controls but focus on a smooth (absolutely continuous) issuance policy, denoted by G. It is optimal in their no commitment model for a firm to *always* issue debt and hence $G{>}0$ always. By contrast, if a firm in our no commitment model is able to borrow, the optimal debt issuance policy controls the firm's total coupon rate so that it always equals $\gamma^* M_t$.

DeMarzo and He (2021)'s smooth debt issuance policy follows from their claim that the value of equity given any total coupon rate C' must be at least as high as the value of equity obtained by changing the total coupon rate to C, $C \lesseqgtr C'$:

$$\mathcal{V}(Y, c') \geq \mathcal{V}(Y, C) + \frac{\mathcal{P}(Y, C)}{C}(C - C') = \mathcal{V}(Y, C) - \frac{\mathcal{P}(Y, C)}{C}(C' - C) \tag{26}$$

because equity holders always have the option without incurring transaction costs to issue or buy back debt at the price $\frac{\mathcal{P}(Y,C)}{C}$ per unit of coupon.[22]

If the inequality (26) holds strictly, DeMarzo and He (2021) demonstrate that the optimal issuance policy is continuous (by their Lemma 1) and they then restrict their attention to smooth policies as opposed to a nonsmooth policy like ours. Assuming equity holders find it optimal to adjust debt smoothly, maximizing their HJB equation gives that

[22]Inequality (26) corresponds to DeMarzo and He (2021)'s equation (4) after accounting for the fact that a bond in our framework is normalized to have a coupon rate of one per unit of time.

$\mathcal{V}_2(Y,C) = -\frac{\mathcal{P}(Y,C)}{C}$ everywhere in the non-default region.[23] This optimality condition can be anticipated from expression (26).

Restricting attention to the case that a firm's cash flows are lognormal, like we do, DeMarzo and He (2021) rule out any nonsmooth equilibrium in this setting. As before, their proof is predicated on inequality (26) holding.[24]

However, whether or not inequality (26) holds depends critically on creditors' expectations regarding equity holders' future leverage decisions. This arbitrage relation relies on a *given* equilibrium issuance policy. That is, while the price of debt does react to a change in the total coupon rate, it is implicitly assumed that equilibrium issuance *policy* going forward remains the same.

If equity holders' proposed leverage change from C' to C prompts creditors to alter their expectations of future issuance policy, creditors will not accept prices based on the previous issuance policy, thereby invalidating equation (26). For example, if increasing the rate of debt issuance convinces creditors that equity holders will increase the future rate of debt issuance by more than they had expected, creditors will only buy additional debt at a price lower than prevailing under the previous issuance policy reflecting their anticipation that bankruptcy will now occur sooner than previously expected. Alternatively, if equity holders propose to buy back existing debt, any individual debt holder will prefer to hold onto their debt in the hope that others will sell.[25] If all existing debt holders collectively behave in this manner, the firm will only be able to buy back debt at a price corresponding to the

[23]This expression corresponds to DeMarzo and He's optimality condition, their equation (8), again after accounting for the fact that a bond in our framework is normalized to have a coupon rate of one per unit of time.

[24]See DeMarzo and He (2021)'s Lemma A.1 and its proof. In order to rule out nonsmooth Markov perfect equilibria, they impose the requirement that $\mathcal{V}_2(Y,C) = -\frac{\mathcal{P}(Y,C)}{C}$ holds everywhere at multiple steps within their proof.

[25]See Berk and DeMarzo (2020), pages 607-609.

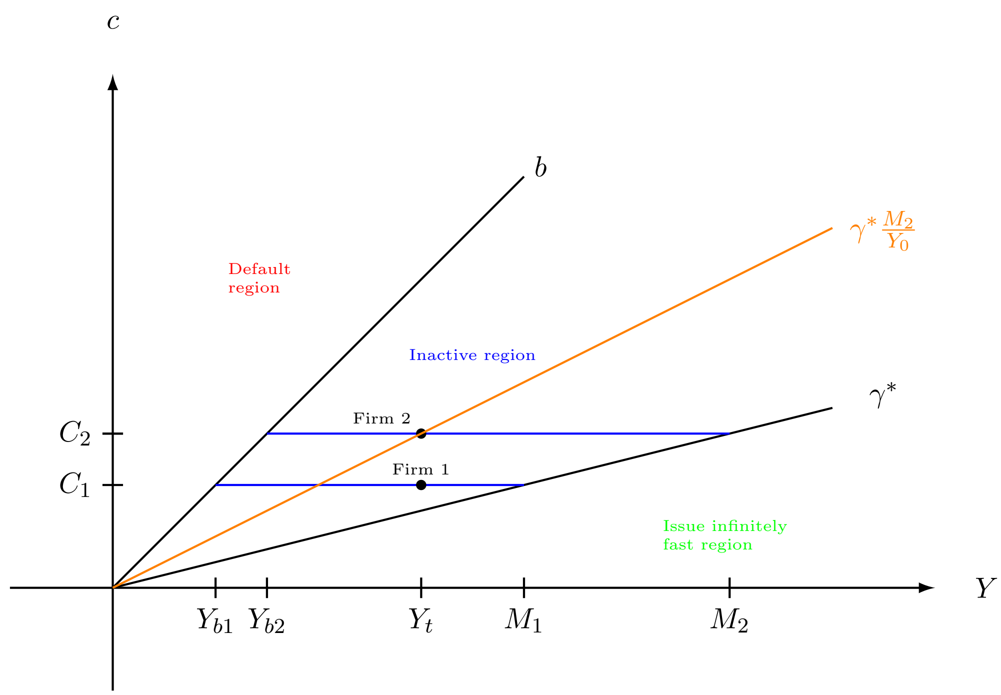


Figure 3: Graphical illustration of apparent arbitrage opportunity.

debt being risk free and not the price prevailing under the firm's issuance policy.

To the extent that a proposed leverage change results in creditors altering their expectations of the firm's future issuance policy, inequality (26) does not hold. As a result, DeMarzo and He (2021) cannot rule out nonsmooth equilibrium, such as ours, when a firm's cash flows are lognormal.

A numerical example using our valuation framework and its solutions for a firm's total debt and equity values will further clarify these issues. Consider the two firms illustrated in Figure 3. The firms are identical save for the fact that Firm 2 has more debt than Firm 1, $C_2 > C_1$, because Firm 2 is older and at some time in the past experienced a higher maximum EBIT, M_2, when compared to the younger Firm 1 whose maximum EBIT is only $M_1 < M_2$. We once again rely on the parameter values assumed by DeMarzo and He (2021) in the case where firm cash flows follow a lognormal process: $r = 5\%$, $\sigma = 40\%$, $\mu = 2\%$, $\tau_i = 0\%$, $\tau_e = 30\%$, $\xi = .2$, $k = 0$ and $\alpha = 100\%$. Furthermore, both firms' current EBIT is $Y_t = 2$, while $M_1 = 2.2$ and $M_2 = 2.8$.

The optimal debt issuance parameter without commitment in this case is $\gamma^* = 0.551$, Firm 2's total coupon rate is $C_2 = 2.8 \times .551 = 1.542$ while for Firm 1 we have $C_1 = 2.2 \times .551 = 1.211$. Solving for the equity and bond values of the two firms gives[26]

$$\text{Firm 1}: \ \mathcal{V}(2, 1.211) = 2.2 \times v(\frac{2}{2.2}, .551) = 34.175 \quad \frac{\mathcal{P}(2, 1.211)}{1.211} = \frac{p(\frac{2}{2.2}, .551)}{.551} = 15.230$$
$$\text{Firm 2}: \ \mathcal{V}(2, 1.542) = 2.8 \times v(\frac{2}{2.8}, .551) = 29.248 \quad \frac{\mathcal{P}(2, 1.542)}{1.542} = \frac{p(\frac{2}{2.8}, .551)}{.551} = 15.031$$

and it follows that

$$34.175 = \mathcal{V}(Y_t, C_1) < \mathcal{V}(Y_t, C_2) + \frac{\mathcal{P}(Y_t, C_2)}{C_2}(C_2 - C_1) = 29.248 + 15.031 \times .551 \times .6 = 34.214,$$

thereby violating inequality (26). Hence, it would appear then to be advantageous for the equity holders of Firm 1 to issue additional debt if they are able to do so at Firm 2's debt price of $\frac{\mathcal{P}(Y_t, C_2)}{C_2}$ per unit of coupon.

However, debt holders value both Firm 1's and Firm 2's debt assuming that neither firm will issue additional debt unless the firm's EBIT increases sufficiently so that a new maximum EBIT is achieved. If Firm 1 were to suddenly announce an unexpected debt issuance inside the continuation region, debt holders' rational reaction to this out-of-equilibrium move would be to conclude that the firm has changed its issuance policy (γ^*) to a more aggressive policy ($\gamma^* \frac{M_2}{Y_t}$) illustrated by the orange line in Figure 3 and so are only willing

[26]The numerical solution to the bankruptcy threshold in this case is $y_b = .2365$ and the par value of the bond, which is also the price at which the firm issues debt, is $P = 15.2439$.

to buy the additional debt issued by Firm 1 at a much lower price. The proceeds the equity holders would now receive from issuing the additional debt would be

$$\frac{\mathcal{P}(Y_t, C_2; \gamma^* \frac{M_2}{Y_t})}{C_2}(C_2 - C_1)$$

where we have augmented the notation of the bond price with the relevant issuance strategy parameter. Taking the new bond price into account gives that[27]

$$34.175 = \mathcal{V}(Y_t, C_1) > \mathcal{V}(Y_t, C_2; \gamma^* \frac{M_2}{Y_t}) + \frac{\mathcal{P}(Y_t, C_2; \gamma^* \frac{M_2}{Y_t})}{C_2}(C_2 - C_1) = 32.016$$

and there is now no incentive for Firm 1's equity holders to deviate from the strategy of not issuing additional debt inside the continuation region.

In our model without commitment, DeMarzo and He (2021)'s requirement that $\mathcal{V}_2(Y, C) = -\frac{\mathcal{P}(Y,C)}{C}$ is violated everywhere inside the continuation region where $Y{<}M$.[28] But this equality does hold when the firm actively issues debt (when $g{>}0$) at the upper boundary where $Y{=}M$ and the total coupon rate satisfies $C{=}\gamma Y$. To see this, using equations (23a)

[27] Recalculating for an issuance policy parameter of $\gamma = \gamma^* \frac{2.8}{2} = .771$ gives $y_b = .302$ and $P = 12.063$. The new equity value is $\mathcal{V}(2, 1.542; .771) = 2 \times v(1, .771) = 28.030$ and the new bond value is $\frac{\mathcal{P}(2,1.542;.771)}{1.542} = \frac{p(1,.771)}{.771} = 12.063$. Hence,

$$\mathcal{V}(Y_t, C_2; \gamma^* \frac{M_2}{Y_t}) + \frac{\mathcal{P}(Y_t, C_2; \gamma^* \frac{M_2}{Y_t})}{C_2}(C_2 - C_1) = 28.030 + 12.063 \times .551 \times .6 = 32.016.$$

[28] For example, continuing our numerical example, we have

$$\mathcal{V}_2(2, 1.211) = \frac{1}{.551}(v(\frac{2}{2.2}, .551) - \frac{2}{2.2}v'(\frac{2}{2.2}, .551)) = -15.117 \neq -\frac{\mathcal{P}(2, 1.211)}{1.211} = -15.230$$
$$\mathcal{V}_2(2, 1.542) = \frac{1}{.551}(v(\frac{2}{2.8}, .551) - \frac{2}{2.8}v'(\frac{2}{2.8}, .551)) = -14.690 \neq -\frac{\mathcal{P}(2, 1.542)}{1.542} = -15.031.$$

The fact that for these specific parameter values $\mathcal{V}_2(Y, C) > -\frac{\mathcal{P}(Y,C)}{C}$ is consistent with the apparent arbitrage that equity holders could exploit by issuing additional debt *if* they were able to do so at the given equilibrium debt price. However, as we have argued, this neglects the fact that issuing additional debt changes the issuance policy parameter, γ, and, as a result, debt holders will lower the price at which they would buy the newly issued debt and thereby nullify the arbitrage opportunity.

and (19), we have

$$\mathcal{V}_2(Y,\gamma Y) = \frac{1}{\gamma}(v(1,\gamma) - v'(1,\gamma)) = \frac{1}{\gamma}(v(1,\gamma) - v(1,\gamma) - (1-k)p(1,\gamma))$$
$$= -(1-k)\frac{p(1,\gamma)}{\gamma} = -(1-k)\frac{\mathcal{P}(Y,\gamma Y)}{\gamma Y}$$

and the desired result follows given DeMarzo and He (2021)'s assumption that k=0 and given that C=γY at the upper boundary.

The optimal issuance policy in our model without commitment maximizes the value of the firm's equity with respect to the issuance policy parameter γ and not with respect to the state variable C. That is, when optimizing their issuance policy, equity holders internalize debt holders' reaction to what price they would be willing to buy the debt the firm issues. Therefore, the relevant first order condition requires evaluation of

$$\frac{d}{d\gamma}\mathcal{V}(Y,C(\gamma)) = \frac{d}{d\gamma}\mathcal{V}(Y,\gamma M) = \frac{d}{d\gamma}(Mv(\frac{Y}{M},\gamma)) = M\frac{d}{d\gamma}v(y,\gamma)$$

where the parameter γ is varied for a given fixed Y and M.[29] To more clearly see the γ dependence in the first order condition characterizing the optimal issuance policy, equation (6), rewrite the equation as

$$(1-k)\frac{\mathcal{P}(M,\gamma M)}{\gamma M} + \mathcal{V}_2(M,\gamma M) = 0$$

[29]When applying our closed form solution for $v(y,\gamma)$, equation (25), it is important to take into account the fact that y_b and P are actually functions of the issuance policy parameter γ. Even though we need to solve for y_b numerically, the equations determining y_b and P (equations (22b) and (7)) are both explicit equations and so the functional forms of $y_b'(\gamma)$ and $P'(\gamma)$ can be derived in closed form by the implicit function theorem.

and applying the homogeneity property from equations (14a) and (14b), gives[30]

$$(1-k)\frac{p(1,\gamma)}{\gamma}+\frac{d}{d\gamma}v(1,\gamma)=0. \tag{27}$$

It is the left hand side of this equation, for example, that we evaluate as a function of γ to derive Figure 2.

3.4 Out-of-Equilibrium Responses

As we have just seen, equity holders in certain situations may have an incentive to deviate from our proposed issuance policy by issuing incremental debt *if* the additional debt could be sold at a price assuming an issuance policy in which equity holders do not issue additional debt until M increases. However, this incentive disappears if the price of debt is adjusted to reflect the fact that lenders interpret any out-of-equilibrium issuance as evidence of a change in the firm's issuance policy. To complete the specification of our model, we extend our debt and equity price functions in Appendix B to account for potential out-of-equilibrium actions by equity holders.

4 Leverage Dynamics

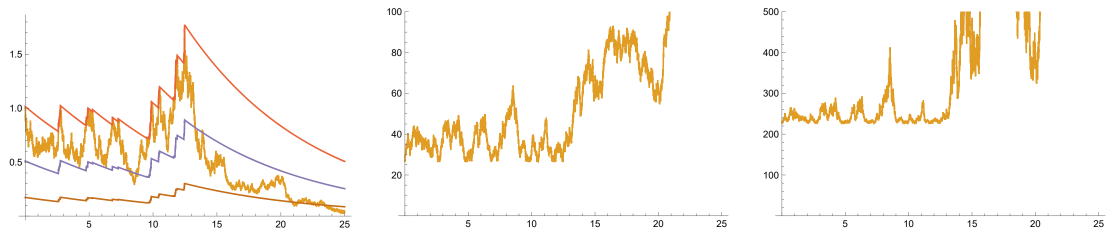
To illustrate the leverage dynamics and other properties of our risky debt model, we continue to rely upon DeMarzo and He (2021)'s base case set of parameter values. Assuming a firm's assets in place generate operating cash flows at a rate governed by geometric Brown-

[30]To see that $\mathcal{V}_2(M,\gamma M)=\frac{d}{d\gamma}v(1,\gamma)$ note that

$$\frac{d}{d\gamma}\mathcal{V}(M,\gamma M))=M\frac{d}{dC}\mathcal{V}(M,C)|_{C=\gamma M}=M\mathcal{V}_2(M,\gamma M)$$

and therefore

$$\mathcal{V}_2(M,\gamma M)=\frac{1}{M}\frac{d}{d\gamma}\mathcal{V}(M,\gamma M)=\frac{d}{d\gamma}v(1,\gamma).$$



(a) State variables. (b) Percentage leverage ratio (c) Credit spread (bps)

Figure 4: Panel (a) displays EBIT Y_t (orange curve), M_t (red curve), the total coupon rate, C_t (purple curve), and the bankruptcy trigger level (brown curve). Panel (b) displays the corresponding percentage leverage ratio (debt value divided by total firm value). Panel (c) displays the debt's credit spread in basis points. The parameters used for the simulation are $r = 5\%$, $\sigma = 40\%$, $\mu = 2\%$, $\tau_i = 0\%$, $\tau_e = 30\%$, $\alpha = 100\%$, $\xi = .1$, and $k = 0$.

ian motion with drift μ=2% and volatility σ=40%, Panel (a) of Figure 4 displays a realized sample path of the firm's EBIT (orange curve) normalized to equal unity at $t = 0$.

Assuming debt with an average maturity of 10 years (ξ=0.1), the firm's optimal debt issuance parameter without commitment is γ^*=0.503, resulting in the firm issuing debt at origination t=0 with a total coupon rate of C_0=0.503. Its leverage ratio at origination, defined as the ratio of the value of debt to the sum of the values of debt and equity, is 26.71%, see Panel (b) of Figure 4, while the credit spread on the debt at origination is 229 bps, see Panel (c) of Figure 4.

In addition to the realized sample path of the firm's EBIT, Panel (a) of Figure 4 also displays the corresponding sample path of the firm's maximum cash flow rate net of the required sinking fund payments, M_t (red curve). The sample path of the firm's total coupon rate, C_t is also displayed in Panel (a) of Figure 4 (purple curve) and is simply the maximum cash flow rate scaled by the firm's optimal debt issuance parameter γ^*. When the firm's realized cash flow rate surpasses the prevailing maximum cash flow rate, the firm incrementally issues debt as can be seen in the resultant increase in the firm's total coupon rate. If cash flows do not attain a new maximum, however, the firm does not issue debt and the firm's existing debt gradually matures according to its sinking fund schedule.

The implied bankruptcy trigger is also displayed in Panel (a) of Figure 4 (brown curve). Notice that the firm's EBIT (orange curve) crosses the bankruptcy trigger (brown curve) at approximately t=20.5 years at which point the firm defaults.

The firm's cash flow dynamics as well as its decision whether or not to incrementally issue debt affects its leverage ratio and the credit spread on its debt. From Panel (b) of Figure 4, we see that when the firm issues debt, its leverage ratio reverts to the ratio prevailing at the firm's origination. The firm's leverage ratio in our model, however, can never be lower than its leverage ratio at origination as equity holders always find it optimal to issue incremental debt when cash flows increase sufficiently. As can be seen in Panel (c) of Figure 4, the credit spread on the firm's debt after issuing debt also reverts to the credit spread prevailing at the firm's origination and cannot be narrower than the spread at the firm's origination. Otherwise, when the firm's cash flows are decreasing or are at least not increasing sufficiently to trigger the issuance of incremental debt, the firm's leverage ratio increases and the credit spread on its existing debt widens, reflecting a higher likelihood of default.[31]

4.1 Optimal Debt Maturity

Unlike DeMarzo and He (2021), our model admits an optimal debt maturity structure. When issuing debt, a firm's optimal debt maturity is determined by finding that value of ξ which maximizes total firm value at $t = 0$:

$$\xi^* = \underset{\xi}{\operatorname{argmax}}\ v(1,\gamma(\xi)) + (1-k)p(1,\gamma(\xi)). \tag{28}$$

[31] Leverage ratios and credit spreads after the firm defaults should be ignored.

The optimization here must account for the fact that corresponding to each possible maturity ξ is an optimal debt issuance policy parameter γ. Having determined the optimal γ for each maturity ξ, we choose from amongst these that maturity which maximizes total firm value. This optimal debt maturity depends on all of the model parameters including prevailing tax provisions as well as proportional issuance costs k.

Equity holders, all else equal, prefer shorter maturity debt. Shorter maturity debt leads to higher leverage to take advantage of debt's tax benefits while the resultant leverage can be rolled over quickly because of the shorter maturity. In the absence of issuance costs, it would be optimal for a firm to be 100% financed with infinitesimally short term debt thereby maximizing tax benefits while eliminating the possibility of bankruptcy and its attendant costs. Issuance costs, however, dampen equity holders enthusiasm to issue debt too aggressively.

Table 1 investigates the effects of issuance costs in our model on a firm's optimal debt maturity and corresponding optimal leverage. Both the commitment and no commitment cases are considered. The underlying parameters are, as before, r=5%, μ=2%, σ=40%, τ_i=0%, τ_e=30%, and α=50%. We now also vary volatility, σ=35% and σ=45%, and introduce proportional issuance costs of $k=\frac{1}{2}$% and k=1%.

We see from Table 1 that with commitment, all else being equal, the firm's optimal debt maturity is longer, optimal leverage is lower, and the credit spread on the firm's debt at issuance is correspondingly narrower than without commitment. As expected, the firm's debt policy is always less aggressive with commitment.

In both the commitment and no commitment cases, increasing issuance costs k, holding volatility σ fixed, lengthens the optimal maturity of the firm's debt, lowers its optimal leverage and concomitantly widens the credit spread on the firm's debt at issuance. For example, assuming equity holders follow a debt policy without commitment, increasing issuance costs from k=$\frac{1}{2}$% to k=1% for σ=45%, lengthens optimal maturity from approximately 1 year to 1.8 years, decreases optimal leverage from 54% to 46% and doubles the credit spread on the firm's debt at issuance from 54 bps to 108 bps. The firm is also seen to issue debt less aggressively when issuance costs increase, γ^*=0.81 at k=$\frac{1}{2}$% to γ^*=0.70 at k=1%. The conclusion that the firm issues debt less aggressively when issuance costs increase holds both in the commitment as well as no commitment cases.

Higher volatility σ, holding issuance costs k fixed, is seen from Table 1 in both the commitment and no commitment cases to shorten the optimal maturity of the firm's debt and decrease its optimal leverage. The credit spread on the firm's debt at issuance widens and the aggressiveness with which the firm issues debt, with or without commitment, is seen to dampen in the face of higher volatility. For example, if equity holders commit to a debt policy, increasing volatility from σ=35% to σ=45% for issuance costs of k=1%, shortens optimal maturity from approximately 2.8 years to 2.6 years, decreases optimal leverage from 42% to 32% and widens the credit spread on the firm's debt at issuance from 30 bps to 51 bps. The firm issues debt more aggressively at σ=35%, γ^{**}=0.62 than at σ=45%, γ^{**}=0.46.

5 Commit or No Commit

As discussed earlier, high yield indentures typically include covenants by which the issuer commits to limit the issuance of additional debt subject to negotiated carveouts of these restrictions designed to preserve the issuer's ability to pay principal and interest on the existing debt. Investment grade indentures, by contrast, typically do not restrict the issuance of additional debt and, as such, the issuer is not committed to a predetermined debt issuance policy.

We now apply our model to investigate why equity holders of a high yield firm would contractually agree to such a debt issuance policy while equity holders of an investment grade firm typically do not. We demonstrate that if high yield issuers did not commit, not only would their firm values fall, potentially precipitously, but creditors would demand far higher rates of interest or, worse still, would otherwise refuse to lend to them. While investment grade issuers could increase firm value by committing to a debt policy, by not doing so they avoid the costs of writing necessary indenture provisions and are also able to lever more.

We focus our attention to ten year U.S. corporate bond spreads reported by Standard and Poor's as of December 31, 2022. In particular, the ten year AAA credit spread stood at 58 basis points, the ten year BB credit spread at 3.08% while the high yield (HY) spread, for U.S. corporate bonds rated C and below, was much wider at 11.70%.

Since AAA rated corporate bonds are clearly investment grade, we assume issuers able to command a AAA rating do not commit to any debt issuance policy. Because BB and HY are not investment grades, these issuers, by contrast, are assumed to have committed to a debt issuance policy.

We next use our model to investigate the counterfactual of the investment grade issuer deciding, alternatively, to commit to a debt issuance policy and the counterfactual of the non-investment grade issuer deciding, alternatively, to not commit. Doing so allows us to gauge the implications of an issuer's decision to commit or not commit.

In particular, we calibrate our model at a ten year maturity (ξ=.10) to fit an observed credit spread, assuming no commitment in the investment grade case and commitment otherwise, by varying the volatility of the firm's operating cash flow rate, σ, for alternative expected operating cash flow growth rates, $\mu = +2\%$ (positive growth), μ=0% (no growth), and $\mu = -2\%$ (negative growth).[32] Once calibrated for an assumed debt issuance policy, we use the resultant fitted parameter values to solve our model for the corresponding counterfactual debt issuance policy, commitment in the investment grade case and no commitment otherwise.

The results are tabulated in Table 2. We see that a AAA-rated firm whose operating cash flows are expected to subsequently grow ($\mu = +2\%$) would increase its value by 2.47% by committing to a debt issuance policy while decreasing the credit spread on its debt at issuance from 58 bps to only 25 bps. However, given the less aggressive debt issuance policy, the AAA-rated firm's leverage ratio at issuance would decrease to 58.62% from 64.04%. Alternatively, a B- or HY-rated firm whose operating cash flows are expected to subsequently grow but does not commit to a debt issuance policy could lever more, a 34.80% *vs* 24.26% leverage ratio at issuance for a BB-rated firm and a 22.48% *vs* 17.74% leverage ratio at issuance for a HY-rated firm, but would do so at much wider credit spreads, 565 bps *vs* 308 bps for a BB-rated firm and 1464 bps *vs* 1170 bps for a HY-rated firm, and experience a decrease in firm value, -4.81% for a BB-rated firm and -1.78% for a HY-rated firm.

If a firm's operating cash flows are not expected to subsequently grow (μ=0% or $\mu = -2\%$), our model cannot be calibrated under no commitment to fit the observed AAA credit spread of 58 basis points. As a result, we cannot in these cases investigate the implications of a AAA rated firm committing to a debt issuance policy.

By contrast, we can calibrate our model with commitment to fit both the observed BB and HY credit spreads if a firm's operating cash flows are not expected to subsequently grow

[32] We also assume r=5%, τ_i=0%, τ_e=30%, and α=50%. Issuance costs are ignored k=0%.

(μ=0% or μ=−2%). However, in the case of a BB rated firm, at the volatilities needed to fit the observed credit spread of 308 bps with commitment, σ=46.4% for μ=0% and σ=44.3% for $\mu = -2$%, the first order condition corresponding to a no commitment debt issuance policy has no solution, implying that creditors would not lend to the BB rated firm without commitment.[33] But creditors would lend to a HY rated firm without commitment at the volatilities needed to fit the observed 1170 bps credit spread with commitment, σ=81.5% for μ=0% and σ=79.3% for $\mu = -2$%. Intuitively, volatility needs to be sufficiently high so that the firm's upside is large enough before creditors find it credible that equity holders would not have the incentive to issue debt so aggressively as to ultimately bankrupt the firm and render lenders' claims worthless. In this case, a HY-rated firm whose operating cash flows are not expected to subsequently grow and which chooses not to commit to a debt issuance policy could lever more, a 34.80% *vs* 24.26% leverage ratio at issuance for μ=0% and a 22.48% *vs* 17.74% leverage ratio at issuance for $\mu = -2$%, but at much wider credit spreads at issuance, now 1781 bps for μ=0 and 2254 bps for $\mu = -2$% and the firm's value would decrease significantly, -4.33% for μ=0 and -8.60% for $\mu = -2$%.

6 Conclusions

The current price of a firm's debt depends on a variety of factors including the firm's future debt issuance policy. Dynamic capital structure models differ widely in how a firm's debt issuance policy is modeled, ranging from prohibiting outright the issuance of additional debt to, at the other extreme, always smoothly issuing incremental debt.

This paper provides a dynamic capital structure model with a debt issuance policy consistent with how incremental debt is actually issued. In particular, in our model it is optimal

[33] The decrease in firm value reported in these cases is obtained by comparing the levered firm value with commitment to the unlevered firm value without commitment.

for a firm to borrow incrementally only when triggered by a sufficiently large increase in its interest coverage ratio and never before. Commitment to this debt issuance policy corresponds to an agreement between equity holders and lenders to include a carveout to a covenant limiting additional debt which is triggered whenever the firm's interest coverage ratio reaches a contractually specified level.

We recognize that in the absence of commitment creditors will not act passively in the face of equity holders' decision to recapitalize. Therefore, recapitalizations in our model may affect how creditors value a firm's debt by prompting creditors to alter their expectations of the firm's debt issuance policy going forward. In fact, if equity holders propose to issue debt extremely aggressively, lenders may not lend to the firm at all.

In our tradeoff model, a firm can, in general, extract tax benefits from issuing debt and an optimal maturity typically exists. With or without commitment, issuance costs matter. Increasing proportional issuance costs, holding all other model parameters constant, lengthens optimal maturity, lowers the optimal leverage ratio, and widens the credit spread on issued debt.

Appendix A

It is optimal under no commitment for equity holders to instantaneously issue debt so that the resultant total coupon rate is given by $C = \gamma^* M$ where γ^* satisfies the following first order condition (equation (27)):

$$(1-k)\frac{p(1,\gamma)}{\gamma} + \frac{d}{d\gamma}v(1,\gamma) = 0.$$

We now analyze the left hand side of this first order condition as a function of γ in more detail. It should be emphasized, as discussed in Section 3.1.3, that this first order condition ignores lenders' reactions to equity holders' incentives to potentially deviate from the proposed issuance policy. Lenders' reactions to equity holders' potentially out-of-equilibrium deviations from the proposed issuance policy are investigated in Appendix B.

As a first observation, the expression

$$(1-k)\frac{p(y,\gamma)}{\gamma}$$

is positive but decreasing in γ. Since $\frac{p(y,\gamma)}{\gamma}$ is the value of a bond with a coupon rate of one per unit time, it clearly will always be positive. As the firm issues debt more aggressively, this value decreases because debt is issued *pari passu* and default occurs sooner as the default trigger, y_b, increases in γ.

As a second observation, the limit of the second expression of the left hand side of the first order condition is

$$\lim_{\gamma\to 0}\frac{d}{d\gamma}v(1,\gamma) = \frac{1}{x_1-1}(1-k)P - (1+\frac{1}{x_1-1})\frac{1-\tau_e+\xi P}{r+\xi}.$$

Here we have used the fact that y_b converges to zero as γ converges to zero and that $\lim_{\gamma\to 0} y_b'(\gamma) = 0$. Moreover, as γ converges to zero, the principal of the debt converges to its risk free value $\frac{1-\tau_i}{r}$. Therefore,

$$\lim_{\gamma\to 0}\frac{d}{d\gamma}v(1,\gamma) = \frac{x_1(\tau_e-\tau_i)}{(x_1-1)(r+\xi)} - \frac{(x_1-(1-k))(1-\tau_i)}{(x_1-1)r}.$$

Combining these results gives

$$\begin{aligned}\lim_{\gamma\to 0}\big((1-k)\frac{p(y,\gamma)}{\gamma}+\frac{d}{d\gamma}v(1,\gamma)\big) &= \frac{x_1(\tau_e-\tau_i)}{(x_1-1)(r+\xi)} - \frac{kx_1(1-\tau_i)}{(x_1-1)r} \\ &= \frac{x_1}{(x_1-1)}\left(\frac{\tau_e-\tau_i}{r+\xi}-\frac{k(1-\tau_i)}{r}\right).\end{aligned}$$

This result tells us where equity holders' first order condition as plotted in Figure 2 commences for $\gamma = 0$. When $k = 0$, the value of this first order condition at $\gamma = 0$ is positive. However, since $\frac{\tau_e-\tau_i}{r+\xi} < \frac{1-\tau_i}{r}$, the first order condition at $\gamma = 0$ will become negative for a sufficiently large $k < 1$. That is, for low issuance costs here, it is beneficial for equity holders to issue debt without commitment but it is not for sufficiently large issuance costs.

The subsequent behavior of equity holders' first order condition as γ increases relies on our third observation that

$$\lim_{\gamma\to\infty}\frac{d}{d\gamma}v(1,\gamma) = 0.$$

Numerical calculations reveal that $\frac{d}{d\gamma}v(1,\gamma)$ is decreasing in γ for low levels of γ and subsequently becomes negative. But at a certain level of γ prior to default, the expression commences increasing but remains negative. Taken together, differences in this behavior as γ increases, depending on the model's parameter values, differentiate whether equity holders' first order condition eventually becomes negative as in Panel (b) in Figure 2 from Panel (c) in Figure 2 where it remains positive throughout.

Appendix B

We now extend our debt and equity value functions to accommodate the possibility of out-of-equilibrium debt issuance by a firm's equity holders. This requires that we take into account lenders' reactions to equity holders' proposed financing decisions. To do so, in addition to observing the prevailing values of the state variables Y and C, we also assume that lenders are able to determine whether equity holders are issuing additional debt. That is, lenders can discern whether $g = 0$ or $g > 0$. This additional assumption is consistent with the fact that firms discuss their financing plans with potential lenders and announce their plans for issuing additional debt.

Define FOC(γ) as the equity holders' first order condition given by the left hand side of equation (27)

$$\text{FOC}(\gamma) = (1-k)\frac{p(1,\gamma)}{\gamma} + \frac{d}{d\gamma}v(1,\gamma).$$

The Case with Debt

To begin with, we assume that the firm is able to issue debt without commitment and focus on the case displayed in Panel (b) of Figure 2 in which bankruptcy costs are realistically assumed to be less than 100%. This requires that FOC(0) > 0 and that there exists a γ^* such that FOC(γ^*) $= 0$ and FOC$'(\gamma^*) < 0$. Typically, for $\gamma > \gamma^*$ sufficiently large, it will be the case that FOC(γ) > 0. If this is the case then we denote the second solution to the first order condition as γ_b which satisfies FOC(γ_b) $= 0$ and FOC$'(\gamma_b) > 0$. That is, γ^* is the left most point where the equity holders' first order curve crosses the horizontal axis and γ_b is the point of the second (upward) crossing of the horizontal axis, as in Panel (b) of Figure 2.[34]

[34]For example, $\gamma^* = 0.848$ and $\gamma_b = 1.588$ in the no commitment case in Panel (b) of Figure 2.

We extend the value function of issued debt in this case to cover all values of $Y \geq 0$ and $C \geq 0$ as follows:

$$\mathcal{P}(Y,C) = \begin{cases} (1-\alpha)\frac{(1-\tau_e)Y}{r-\mu} & \text{if } \frac{Y}{C} < \frac{y_b}{\gamma^*} \text{ and } g = 0 \\ \frac{C}{\gamma^*}p(\frac{\gamma^* Y}{C}, \gamma^*) & \text{if } \frac{y_b}{\gamma^*} \leq \frac{Y}{C} \leq \frac{1}{\gamma^*} \text{ and } g = 0 \\ \frac{C}{\gamma^*}p(1,\gamma^*) & \text{if } \frac{1}{\gamma^*} < \frac{Y}{C} \text{ and } g = 0 \\ 0 & \text{if } \frac{Y}{C} < \max\{\frac{y_b}{\gamma^*}, \frac{1}{\gamma_b}\} \text{ and } g > 0 \\ Yp(1, \frac{C}{Y}) & \text{if } \max\{\frac{y_b}{\gamma^*}, \frac{1}{\gamma_b}\} \leq \frac{Y}{C} \leq \frac{1}{\gamma^*} \text{ and } g > 0 \\ \frac{C}{\gamma^*}p(1,\gamma^*) & \text{if } \frac{1}{\gamma^*} < \frac{Y}{C} \text{ and } g > 0. \end{cases}$$

According to the first expression, when $y = \frac{Y}{M} = \frac{\gamma^* Y}{C}$ is less than y_b, the firm is bankrupt and debt holders take over the firm net of bankruptcy costs. The second expression is the equilibrium value of debt inside the continuation region $y \in [y_b, 1]$. The third and sixth expressions reflect the fact that debt is selling at par at the upper boundary whether or not the firm is issuing new debt. Debt holders here expect that the firm will issue debt up to $C = \gamma^* Y$ and therefore the price per unit of coupon reflects this and sells for $\frac{p(1,\gamma^*)}{\gamma^*}$ which is the par value of the debt P. The fourth expression accounts for the case where equity holders attempt to issue debt so aggressively that their issuance policy parameter exceeds γ_b or when they attempt to issue debt even after the firm is already bankrupt. Lenders can detect equity holders' issuance policy as they observe that the firm is attempting to issue debt ($g > 0$) and by calculating $\gamma = \frac{C}{Y}$. Therefore, lenders foresee that equity holders would be willing to issue new debt at any positive price per unit of coupon and that the firm will consequently be driven to immediate bankruptcy. Thus the only rational reaction from the lenders' point of view would be to set the price of debt to zero.[35] Finally, the

[35]This then leads to the result that the price of all debt must also be zero even though there is still the

fifth expression captures the situation in which equity holders attempt to issue debt so aggressively that their issuance policy parameter exceeds γ^* but not so aggressively that it also exceeds γ_b. Again, lenders can calculate equity holders' issuance policy parameter $\gamma = \frac{C}{Y}$ and so lenders price the debt assuming that $\frac{C}{Y}$ is the new debt issuance policy parameter.

For completeness, we also extend the price of one unit of debt, $\frac{\mathcal{P}(Y,C)}{C}$, to make precise how this price behaves in the limiting cases as C goes to either zero or infinity. Relying on the previous expressions, we have:

$$\frac{\mathcal{P}(Y,C)}{C} = \begin{cases} (1-\alpha)\frac{(1-\tau_e)Y}{(r-\mu)C} & \text{if } \frac{Y}{C} < \frac{y_b}{\gamma^*} \text{ and } g = 0 \\ \frac{1}{\gamma^*}p(\frac{\gamma^* Y}{C}, \gamma^*) & \text{if } \frac{y_b}{\gamma^*} \leq \frac{Y}{C} \leq \frac{1}{\gamma^*} \text{ and } g = 0 \\ \frac{1}{\gamma^*}p(1,\gamma^*) & \text{if } (\frac{1}{\gamma^*} < \frac{Y}{C} \text{ and } g = 0) \text{ or C=0} \\ 0 & \text{if } \frac{Y}{C} < \max\{\frac{y_b}{\gamma^*}, \frac{1}{\gamma_b}\} \text{ and } g > 0 \\ \frac{Y}{C}p(1,\frac{C}{Y}) & \text{if } \max\{\frac{y_b}{\gamma^*}, \frac{1}{\gamma_b}\} \leq \frac{Y}{C} \leq \frac{1}{\gamma^*} \text{ and } g > 0 \\ \frac{1}{\gamma^*}p(1,\gamma^*) & \text{if } (\frac{1}{\gamma^*} < \frac{Y}{C} \text{ and } g > 0) \text{ or C=0.} \end{cases} \tag{29}$$

Note that the limiting case of C going to infinity in the first expression corresponds to a bankrupt firm that in the past experienced an extremely high EBIT Y so that $M \approx \infty$. The price of one unit of debt will therefore be approximately zero at bankruptcy as there will be approximately an infinite number of bond holders to share the bankruptcy proceeds. Also, even if $C = 0$ in the third and sixth expressions, the firm's debt will not be priced at its risk free value of $\frac{1-\tau_i}{r}$ but rather will be priced to reflect the fact that lenders still expect the firm to instantaneously issue debt up to $C = \gamma^* Y$.

value $(1-\alpha)\frac{(1-\tau_e)Y}{r-\mu}$ to be shared among debt holders. But since debt holders fear C will approach infinity, the only rational reaction from an individual lender's point of view is to value debt at a price of zero.

Finally, we also extend the equity value function:

$$\mathcal{V}(Y,C) = \begin{cases} 0 & \text{if } \frac{Y}{C} < \frac{y_b}{\gamma^*} \text{ and } g = 0 \\ \frac{C}{\gamma^*} v(\frac{\gamma^* Y}{C}, \gamma^*) & \text{if } \frac{y_b}{\gamma^*} \leq \frac{Y}{C} \leq \frac{1}{\gamma^*} \text{ and } g = 0 \\ Y v(1,\gamma^*) + (\gamma^* Y - C)\frac{1}{\gamma^*} p(1,\gamma^*) & \text{if } \frac{1}{\gamma^*} < \frac{Y}{C} \text{ and } g = 0 \\ 0 & \text{if } \frac{Y}{C} < \max\{\frac{y_b}{\gamma^*}, \frac{1}{\gamma_b}\} \text{ and } g > 0 \\ Y v(1, \frac{C}{Y}) & \text{if } \max\{\frac{y_b}{\gamma^*}, \frac{1}{\gamma_b}\} \leq \frac{Y}{C} \leq \frac{1}{\gamma^*} \text{ and } g > 0 \\ Y v(1,\gamma^*) + (\gamma^* Y - C)\frac{1}{\gamma^*} p(1,\gamma^*) & \text{if } \frac{1}{\gamma^*} < \frac{Y}{C} \text{ and } g > 0. \end{cases} \tag{30}$$

The first expression reflects the fact that when $y = \frac{Y}{M} = \frac{\gamma^* Y}{C}$ is less than y_b, the firm is bankrupt and equity is worthless. The second expression is the equilibrium value of equity inside the continuation region $y \in [y_b, 1]$. The third and sixth expressions give that at the upper boundary it is optimal for equity holders to issue additional debt up to $C = \gamma^* Y$ at the par value $P = \frac{p(1,\gamma^*)}{\gamma^*}$ per unit of coupon and if C is currently less than $\gamma^* Y$, the price of equity captures the proceeds from the still not yet issued debt. In the fourth expression, equity holders attempt to issue debt so aggressively that their issuance policy parameter exceeds γ_b, leading lenders to set the price per unit of debt to zero in reaction to the prospect of immediate bankruptcy. Hence, there are no proceeds from issuing additional debt and the value of the equity is zero. This expression also covers the case when equity holders attempt to issue debt when the firm is already bankrupt. Finally, the fifth expression reflects the situation where equity holders try to issue debt so aggressively that their issuance policy parameter exceeds γ^*, but still not so aggressively that it also exceeds γ_b. This would lead debt holders to conclude that the issuance policy going forward will be governed by $\gamma = \frac{C}{Y}$ and, as a result, equity will be priced assuming that $\frac{C}{Y}$ is the new debt issuance policy parameter.

The Case with No Debt

The remaining case is where the firm is *not* able to issue debt without commitment; see Panel (c) of Figure 2. This corresponds to either FOC(0) < 0 or, alternatively, FOC(γ) ≥ 0 for all $\gamma \geq 0$. In the former, equity holders do not find it optimal to issue debt because issuance costs are too high. In the latter, equity holders are incentivized to issue debt if there would exist lenders willing to buy the debt at any positive price. However, lenders foresee equity holders' incentives here and realize that if they were to offer to buy the debt at a positive price then equity holders would continue issuing debt, thereby prompting the firm's immediate bankruptcy. Lender's only rational reaction in this case would be to set the price of debt to zero. Therefore, we extend the value function of issued debt as follows:

$$\mathcal{P}(Y, C) = 0.$$

That is, the price of debt is zero either due to the fact that $C = 0$ or because any attempt by equity holders to issue debt will lead to immediate bankruptcy.

For completeness, we also extend the price of one unit of debt, $\frac{\mathcal{P}(Y,C)}{C}$, to make precise how it behaves in the limiting cases as C goes to either zero or infinity:

$$\frac{\mathcal{P}(Y, C)}{C} = 0.$$

The price of one unit of debt is zero even in the case that $C = 0$ because, otherwise, equity holders would be incentivized to continue issuing debt and thereby drive the firm to immediate bankruptcy.

$$
\mathcal{V}(Y,C) = \begin{cases} \frac{1-\tau_e}{r-\mu} & \text{if } C = 0 \text{ and } g = 0 \\ 0 & \text{if } C > 0 \text{ or } C = 0 \text{ and } g > 0. \end{cases}
$$

This expression reflects the fact that the firm either will not or cannot issue debt. If the firm will not issue debt then the equity is priced as the present value of all future earnings without any tax benefits of potential debt financing or bankruptcy costs. Alternatively, any attempt to issue debt in this case will lead to immediate bankruptcy and render the firm's equity worthless.

Incentives to Deviate

We can now establish that our equity holders' equilibrium issuance policy without commitment represents a Markov perfect equilibrium. We continue to consider the case displayed in Panel (b) of Figure 2 in which bankruptcy costs are realistically assumed to be less than 100%.

To begin with, we show that equity holders prefer not to issue additional debt within our model's continuation region. That is,

$$
\mathcal{V}_{g=0}(Y,C_0) \geq \mathcal{V}_{g>0}(Y,C) + (1-k)(C-C_0)\frac{\mathcal{P}_{g>0}(Y,C)}{C} \tag{31}
$$

for all Y and C_0 and all $C > C_0$.[36]

Additionally, we also ensure that equity holders do not have an incentive to buy back debt

[36] For ease of exposition, we now amend the notation for the equity and debt value functions to include a subscript indicating whether the corresponding value is evaluated assuming that equity holders issue debt (g>0) versus do not issue debt (g=0).

at par value. That is,

$$\mathcal{V}_{g=0}(Y, C_0) \geq \mathcal{V}_{g=0}(Y, C) - (C_0 - C)P$$

for all Y and C_0 and all $C < C_0$.

Incentives to deviate by issuing additional debt We first investigate whether equity holders have an incentive to issue additional debt up to a level C where $C > C_0$ for all Y and all C_0. If this were to be the case, then the inequality (31) would be violated.

If $y = \frac{\gamma^* Y}{C_0} < y_b$, we are in the "bankruptcy" region giving that $\mathcal{V}_{g=0}(Y, C_0) = 0$ from the extended definition of equity, equation (30). Moreover, as $\frac{Y}{C} < \frac{Y}{C_0} < \frac{y_b}{\gamma^*}$, it follows that the right hand side of the inequality is also zero from equations (29) and (30). Hence, the inequality holds with equality in this case.

If $C_0 < \gamma^* Y$, we are in the "issue infinitely fast region" and therefore debt holders expect the firm to issue additional debt.

For $C \leq \gamma^* Y$, the inequality above holds with equality. That is,

$$\begin{aligned}
\mathcal{V}_{g=0}(Y, C_0) &= Y v(1, \gamma^*) + (1-k)(\gamma^* Y - C_0)\frac{1}{\gamma^*}p(1, \gamma^*) \\
&= Y v(1, \gamma^*) + (1-k)(\gamma^* Y - C)\frac{1}{\gamma^*}p(1, \gamma^*) + (1-k)(C - C_0)\frac{1}{\gamma^*}p(1, \gamma^*) \\
&= \mathcal{V}_{g>0}(Y, C) + (1-k)(C - C_0)\frac{\mathcal{P}_{g>0}(Y, C)}{C},
\end{aligned}$$

using the extended definitions of debt and equity, equations (29) and (30), respectively.

For $C > \gamma^* Y$, we split the proposed debt issuance into two sequential debt issues. First, the firm issues additional debt up to $C = \gamma^* Y$ which is covered by the preceding case. Now we can "reset" C_0 so that $C_0 = \gamma^* Y$. Since $C > \gamma^* Y = C_0$, the second issuance of debt from $\gamma^* Y$ up to C now occurs inside the continuation region which is the next situation we consider.

We are inside the continuation region when $y = \frac{\gamma^* Y}{C_0} \in [y_b, 1]$ and so the firm in equilibrium should not issue debt. Using the extended definitions of debt and equity, equations (29) and (30), respectively, the inequality (31) can be rewritten as

$$\frac{C_0}{\gamma^*} v(\frac{\gamma^* Y}{C_0}, \gamma^*) \geq Y v(1, \frac{C}{Y}) + (1-k)(C - C_0) \frac{Y p(1, \frac{C}{Y})}{C}.$$

Dividing by Y and substituting $y = \frac{\gamma^* Y}{C_0}$ and $\gamma = \frac{C}{Y}$, we have

$$\frac{1}{y} v(y, \gamma^*) \geq v(1, \gamma) + (1-k)(\gamma - \frac{\gamma^*}{y}) \frac{p(1,\gamma)}{\gamma}. \tag{32}$$

Equity holders prefer not to issue additional debt if this inequality holds for all $y \in [y_b, 1]$ and for all $\gamma > \frac{\gamma^*}{y}$. The right hand side of this expression is zero for $\gamma > \gamma_b$ by equations (29) and (30) and so the inequality is fulfilled in this case as the left hand side of this expression is always non-negative.

Therefore, it only remains to check the inequality (32) for $y \in [y_b, 1]$ and for $\gamma \in [\frac{\gamma^*}{y}, \gamma_b]$. To do so, we rely on the following:

Lemma. *For $\gamma_0, \gamma \in [\gamma^*, \gamma_b]$ with $\gamma > \gamma_0$ we have that*

$$v(1, \gamma_0) \geq v(1, \gamma) + (1-k)(\gamma - \gamma_0) \frac{p(1,\gamma)}{\gamma}. \tag{33}$$

Proof. Using that the first order condition, equation (27), is negative for $\gamma \in [\gamma^*, \gamma_b]$, we have that

$$v_2(1, \gamma) \equiv \frac{d}{d\gamma} v(1, \gamma) \leq -(1-k) \frac{p(1,\gamma)}{\gamma}$$

for $\gamma \in [\gamma^*, \gamma_b]$. By the fundamental theorem of calculus, we have

$$
\begin{aligned}
v(1,\gamma) &= v(1,\gamma_0) + \int_{\gamma_0}^{\gamma} v_2(1,\tilde{\gamma})d\tilde{\gamma} \\
&\leq v(1,\gamma_0) - (1-k)\int_{\gamma_0}^{\gamma} \frac{p(1,\tilde{\gamma})}{\tilde{\gamma}}d\tilde{\gamma} \\
&\leq v(1,\gamma_0) - (1-k)\int_{\gamma_0}^{\gamma} \frac{p(1,\gamma)}{\gamma}d\tilde{\gamma} \\
&= v(1,\gamma_0) - (1-k)(\gamma-\gamma_0)\frac{p(1,\gamma)}{\gamma}
\end{aligned}
$$

for $\gamma_0, \gamma \in [\gamma^*, \gamma_b]$ with $\gamma > \gamma_0$. The second inequality follows by observing that $\frac{p(1,\gamma)}{\gamma}$ is positive and decreasing in γ. Rewriting this result, we have that

$$
v(1,\gamma_0) \geq v(1,\gamma) + (1-k)(\gamma-\gamma_0)\frac{p(1,\gamma)}{\gamma}
$$

for $\gamma_0, \gamma \in [\gamma^*, \gamma_b]$ with $\gamma > \gamma_0$. □

The Lemma demonstrate that a discrete issuance of debt in the continuation region will never be optimal for equity holders. Figure 5 provides an illustration. Here the firm is initiated at $t = 0$ when its EBIT is Y_0 and instantaneously issues debt so that the total coupon rate is $C_0^* = \gamma^* Y_0$. Without loss of generality, assume that Y_0 will also be the firm's highest EBIT level and, for simplicity, ignore the effect of debt maturing due to a sinking fund provision. Assume that the firm's EBIT subsequently falls to Y_j and posit that equity holders find a discrete issuance of debt from C_0^* to C_j^* to be optimal here:

$$
\frac{C_0^*}{\gamma^*}v(\frac{\gamma^* Y_j}{C_0^*},\gamma^*) < Y_j v(1,\frac{C_j^*}{Y_j}) + (1-k)(C_j^* - C_0^*)\frac{Y_j p(1,\frac{C_j^*}{Y_j})}{C_j^*}.
$$

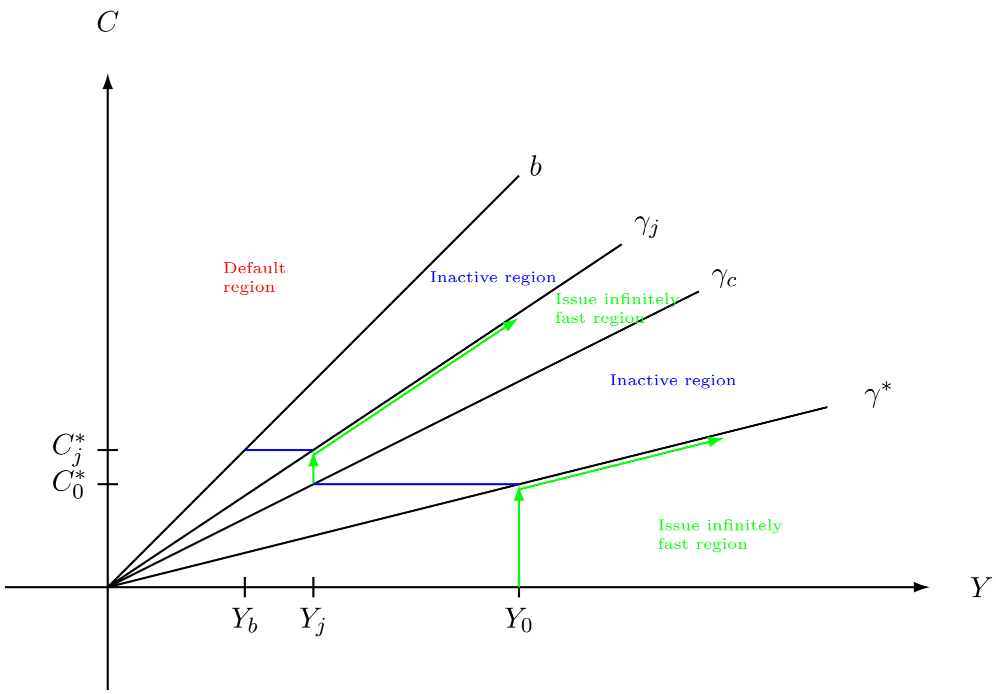


Figure 5: Graphical illustration of a potential model with a discrete issuance inside the continuation region.

However, the Lemma gives that

$$Y_j v(1, \frac{C_0^*}{Y_j}) \geq Y_j v(1, \frac{C_j^*}{Y_j}) + (1-k)(C_j^* - C_0^*)\frac{Y_j p(1, \frac{C_j^*}{Y_j})}{C_j^*}.$$

That is, instead of a discrete debt issuance up to C_j^* and subsequently following the issuance policy $\gamma_j \equiv \frac{C_j^*}{Y_j}$, equity holders are even better of maintaining the firm's coupon rate at C_0^* and updating their issuance policy going forward to $\gamma_c \equiv \frac{C_0^*}{Y_j}$. Notice that this new issuance policy is more aggressive than equity holders' original optimal policy $\gamma^* = \frac{C_0^*}{Y_0}$ but less aggressive than the issuance policy γ_j and, as can be seen in Figure 5, does *not* involve a discrete issuance of incremental debt.

We now use the Lemma to help establish that equity holders prefer not to issue additional debt within our model's continuation region. In doing so, we explicitly recognize that equity holders may switch to the aforementioned more aggressive debt issuance policy if the firm's EBIT declines sufficiently to near bankruptcy, .

Recall that our remaining task is to ensure that the inequality (32)

$$\frac{1}{y}v(y, \gamma^*) \geq v(1, \gamma) + (1-k)(\gamma - \frac{\gamma^*}{y})\frac{p(1, \gamma)}{\gamma}$$

holds for $y \in [y_b, 1]$ and for $\gamma \in [\frac{\gamma^*}{y}, \gamma_b]$. The Lemma gives that

$$v(1, \frac{\gamma^*}{y}) \geq v(1, \gamma) + (1-k)(\gamma - \frac{\gamma^*}{y})\frac{p(1, \gamma)}{\gamma}$$

and so if we can show that

$$\frac{1}{y}v(y, \gamma^*) \geq v(1, \frac{\gamma^*}{y}) \tag{34}$$

for $y \in [y_b, 1]$, we will have also shown that inequality (32) holds. Notice that for $y \leq \frac{\gamma^*}{\gamma_b}$

then $v(1, \frac{\gamma^*}{y}) = 0$ whereas $v(y, \gamma^*) \geq 0$ implying that we only need to show inequality (34) holds for $y \in [\frac{\gamma^*}{\gamma_b}, 1]$. Equivalently, we can re-parameterize this inequality to

$$v(\frac{\gamma^*}{\gamma}, \gamma^*) \geq \frac{\gamma^*}{\gamma} v(1, \gamma) \tag{35}$$

for all $\gamma \in [\gamma^*, \gamma_b]$.

The inequality (35) is satisfied for many assumed parameter values but not all. For example, it is violated for the parameter values underlying Panel (b) of Figure 2: $r = 5\%$, $\mu = 2\%$, $\sigma = 40\%$, $\tau_i = 0\%$, $\tau_e = 30\%$, $\xi = .2$, $k = 0$, and $\alpha = 50\%$. In this case, $\gamma^* = 0.848$ and $\gamma_b = 1.588$ and for $\gamma = 1.5$ we have $v(\frac{\gamma^*}{1.5}, \gamma^*) = 4.114$ while $\frac{\gamma^*}{1.5} v(1, 1.5) = 4.271$ and so violating inequality (35).

For these parameter values, this example suggests that when the firm's EBIT falls sufficiently, equity holders have an incentive to switch to the more aggressive issuance policy $\gamma = 1.5$. However, as we now demonstrate, the implications of implementing this aggressive issuance policy in our valuation framework will dissuade equity holders from deviating to this issuance policy.

To see this, we continue to rely on the previously assumed parameter values and refer to Figure 5 to fix matters. For simplicity, at $t = 0$ assume that $Y_0 = M = 1$. The firm then instantaneously issues debt with a coupon rate of $C_0 = \gamma^* = .848$ at a par value of $\frac{p(1,\gamma^*)}{\gamma^*} = 13.1254$ and the value of equity at origination is $v(1, \gamma^*) = 15.185$. Consistent with our numerical example, assume that the firm's EBIT subsequently declines to $Y_j = \frac{\gamma^*}{1.5} = .566$ and the value of the firm's equity under the original issuance policy is now only $v(\frac{\gamma^*}{1.5}, \gamma^*) = 4.114$. If equity holders here were to switch to the aggressive issuance policy $\gamma_c = \frac{.848}{.566} = 1.5$ then recasting our valuation procedures using $Y_j = M = .566$ and maintaining the current coupon rate of $C_0 = .848$ gives the higher equity value of

$Y_j v(1, 1.5) = 4.271$.

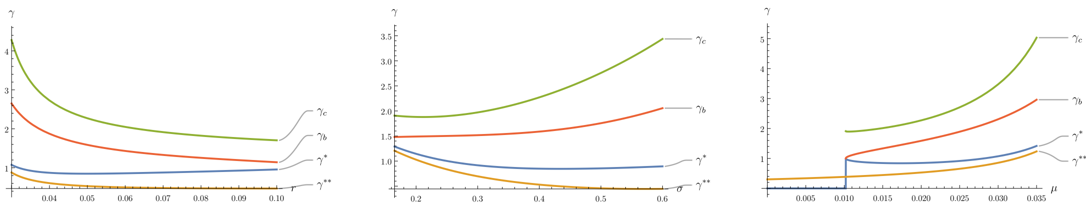
But simply comparing the value of equity after switching to the more aggressive issuance policy to its value prevailing under the issuance policy at origination ignores the fact that all debt issued in our model, initially or incrementally, must be at the same par value, equation (7). Debt issued subsequent to the proposed switch to the more aggressive policy, however, would be issued at a par value lower than the par value determined at the initial debt's issuance. To illustrate, in our numerical example, debt issued after the switch has a par value of $\frac{p(1,1.5)}{1.5} = 7.631$ as compared to the higher par value of $\frac{p(1,\gamma^*)}{\gamma^*} = 13.1254$ when the firm's debt was originally issued. Therefore to be consistent with our valuation framework requires that debt originated pursuant to the more aggressive issuance policy be issued at a discount from the par value determined when the firm's initial debt was issued. Therefore, to properly determine whether equity holders have an incentive to deviate to this more aggressive issuance policy, we extend our valuation framework to take into account the fact that the debt issued at a discount after the proposed switch will be paid back at a higher par value determined when the firm's debt was initially issued.

The value of equity in the extended model is denoted by $v_e(y, \gamma^*; y_c, \gamma)$ where y is the firm's current scaled EBIT, γ^* is the issuance policy determined when the firm's initial debt was issued, y_c is the scaled EBIT where the firm switches to the more aggressive issuance policy, and finally, γ is the new issuance policy after the switch. Equity holders now have no incentive to issue additional debt in our model's continuation region if the following holds:

$$v(\frac{\gamma^*}{\gamma}, \gamma^*) \geq v_e(\frac{\gamma^*}{\gamma}, \gamma^*; \frac{\gamma^*}{\gamma}, \gamma)$$

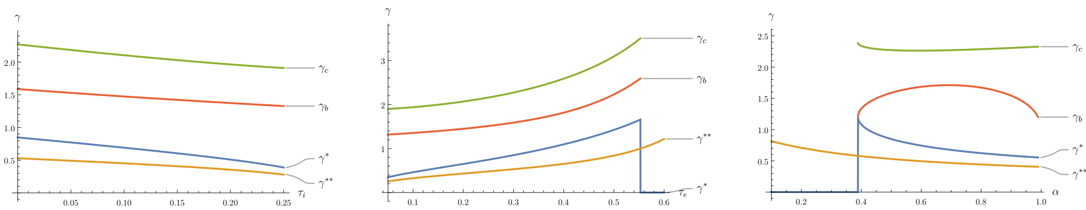
for all $\gamma \in [\gamma^*, \gamma_b]$.

Based on extensive numerical calculations, there appear to be no parameter values of



(a) Interest rate (b) Volatility (c) Drift

Figure 6: Comparative Statics 1



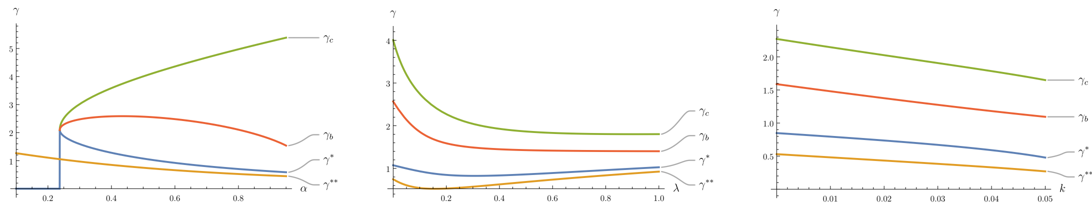
(a) Interest income tax rate (b) Equity tax rate (c) Bankruptcy costs

Figure 7: Comparative Statics 2

our model that violate this expression. We illustrate the robustness of this conclusion in Figures 6, 7, and 8 where we plot for varying values of our model's parameters the smallest value of γ that solves

$$v(\frac{\gamma^*}{\gamma}, \gamma^*) = v_e(\frac{\gamma^*}{\gamma}, \gamma^*, \frac{\gamma^*}{\gamma}, \gamma)$$

and compare this numerically determined value, denoted by γ_c, to the corresponding value of γ_b. Recall that lenders will not lend to the firm for issuance policies more aggressive than γ_b. As can be seen in these Figures, γ_c reliably exceeds γ_b for all reasonable parameter values meaning that equity holders have no incentive to deviate to the more aggressive issuance policy in our model's continuation region. The Figures also display the corresponding optimal issuance policies, both with (γ^{**}) and without commitment (γ^*), and therefore provide additional comparative statics of our model with respect to its parameters.



(a) Bankruptcy costs, infinite maturity

(b) Maturity parameter

(c) Issuance costs

Figure 8: Comparative Statics 3

Incentives to deviate by buying back debt We also need to ensure that equity holders do not have an incentive to buy back debt at its par value. That is,

$$\mathcal{V}_{g=0}(Y, C_0) \geq \mathcal{V}_{g=0}(Y, C) - (C_0 - C)P$$

for all Y and for all C_0 and all $C < C_0$.[37]

To begin with, for $C_0 \leq \gamma^* Y$, we are in the "issue infinitely fast" region and the relation holds here with equality

$$\begin{aligned}
\mathcal{V}(Y, C_0) &= Yv(1, \gamma^*) + (1-k)(\gamma^* Y - C_0)\frac{1}{\gamma^*}p(1, \gamma^*) \\
&= Yv(1, \gamma^*) + (1-k)(\gamma^* Y - C)\frac{1}{\gamma^*}p(1, \gamma^*) - (1-k)(C_0 - C)\frac{1}{\gamma^*}p(1, \gamma^*) \\
&= \mathcal{V}(Y, C) - (C_0 - C)\frac{1}{\gamma^*}p(1, \gamma^*) = \mathcal{V}(Y, C) - (C_0 - C)P
\end{aligned}$$

for $C \leq C_0$ using the extended definition of equity, equation (30), and the assumption that new debt is issued at par, equation (7).

For the more interesting case when $\gamma^* Y < C_0$, using the fundamental theorem of calculus,

[37]Going forward, we omit the subscript $g = 0$ from the equity value function's notation as we are investigating equity holders' incentives to buy back debt at par and there are no strategic considerations on the part of debt holders as to what price to sell back their debt.

we have

$$\mathcal{V}(Y, C_0) = \mathcal{V}(Y, C) + \int_C^{C_0} \mathcal{V}_2(Y, \tilde{C}) d\tilde{C}$$

for $C \in [\gamma^* Y, C_0]$. Note that this relation holds even when $C_0 \geq \frac{\gamma^* Y}{y_b}$, *e.g.*, when C_0 is so large that the firm is bankrupt, if we extend the definition of $\mathcal{V}_2(Y, C)$ so that this derivative is equal to zero for $C \geq \frac{\gamma^* Y}{y_b}$. Because $\mathcal{V}(Y, C)$ is convex in C, that is, the marginal loss of an extra unit of C falls as C increases, for a given fixed Y and a given fixed issuance policy γ[38], we have

$$\begin{aligned}
\mathcal{V}_2(Y, C_0) &\geq \mathcal{V}_2(Y, C) \\
&\geq \mathcal{V}_2(Y, \gamma^* Y) \\
&= \frac{1}{\gamma^*} v(1, \gamma^*) - \frac{1}{\gamma^*} v'(1, \gamma^*) \\
&= \frac{1}{\gamma^*} v(1, \gamma^*) - \frac{1}{\gamma^*} (v(1, \gamma^*) + (1-k) p(1, \gamma^*)) \\
&= -\frac{(1-k)}{\gamma^*} p(1, \gamma^*)) \\
&\geq -\frac{1}{\gamma^*} p(1, \gamma^*)) \\
&= -P
\end{aligned}$$

for $C \in [\gamma^* Y, C_0]$. This calculation relies on equations (19), (23a), and (7). That is,

$$\mathcal{V}(Y, C_0) = \mathcal{V}(Y, C) + \int_C^{C_0} \mathcal{V}_2(Y, \tilde{C}) d\tilde{C} \geq \mathcal{V}(Y, C) - (C_0 - C) P$$

for all Y and for all $C_0 > \gamma^* Y$ and all $C \in [\gamma^* Y, C_0]$.

Taken together, these calculations show that equity holders do not have an incentive to deviate by buying back debt at its par value P.

[38]Note that this means that $-\mathcal{V}_2(Y, C_0) \leq -\mathcal{V}_2(Y, C)$ for $C \in [\gamma Y, C_0]$ as the marginal loss is $-\mathcal{V}_2(Y, \cdot)$.

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Table 1: Proportional Issuance Costs

No Committment:

$k=\frac{1}{2}\%$	$\sigma=35\%$	$\sigma=40\%$	$\sigma=45\%$
maturity	1.22373 years	1.10374 years	1.00478 years
leverage	61.7867%	57.8934%	54.3598%
credit spread	37.188 bps	45.7807 bps	54.8712 bps
γ^*	0.952591	0.876936	0.812929

$k=1\%$	$\sigma=35\%$	$\sigma=40\%$	$\sigma=45\%$
maturity	2.14191 years	1.94799 years	1.80246 years
leverage	53.8066%	49.6988%	45.9393%
credit spread	69.5737 bps	87.2825 bps	107.526 bps
γ^*	0.817032	0.753585	0.701417

Committment:

$k=\frac{1}{2}\%$	$\sigma=35\%$	$\sigma=40\%$	$\sigma=45\%$
maturity	1.38239 years	1.26953 years	1.18457 years
leverage	54.7426%	49.7005%	44.9844%
credit spread	14.4068 bps	17.2332 bps	20.3924 bps
γ^{**}	0.824719	0.730831	0.648749

$k=1\%$	$\sigma=35\%$	$\sigma=40\%$	$\sigma=45\%$
maturity	2.79033 years	2.72468 years	2.60624 years
leverage	42.2228%	36.3393%	31.6281%
credit spread	29.9735 bps	40.0588 bps	50.5693 bps
γ^{**}	0.617312	0.527491	0.45819

Table 2: Fitting prevailing credit spreads

	AAA		BB		HY	
	commit	no commit	commit	no commit	commit	no commit
	25 bps	58 bps	308 bps	565 bps	1170 bps	1464 bps
$\mu = 2\%$	58.62%	64.04%	24.26%	34.80%	17.74%	22.48%
	+2.47%			-4.81%		-1.78%
	-	-	308 bps	NA	1170 bps	1781 bps
$\mu = 0\%$	-	-	26.00%	0%	19.38%	29.09%
	-			-9.33%		-4.33%
	-	-	308 bps	NA	1170 bps	2254 bps
$\mu = -2\%$	-	-	27.18%	0%	20.38%	36.52%
	-			-7.00%		-8.60%