Demandable Debt and Leverage Ratchet Effect*

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Abstract

In this paper, we demonstrate that demandable debt provides an effective solution to the leverage ratchet effect without requiring any additional information beyond that assumed in the existing literature. Demandable debt-holders have an option to request full repayment of debt at any time. If the firm's leverage exceeds its target debt ratio, debt-holders will exercise their option and sell this excess debt back to the firm. This mechanism efficiently disciplines the firm to maintain the target debt ratio, except under extreme negative shocks leading to inevitable bankruptcy. Furthermore, we show that as the model's time intervals shorten, the firm can asymptotically achieve the full tax shield benefits without incurring any bankruptcy risk.

Keywords: Capital Structure, Demandable Debt, Leverage Ratchet Effect

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1 Introduction

Admati, DeMarzo, Hellwig, and Pfleiderer (2018) suggest that when a firm cannot commit to future leverage choices, it tends to resist reductions in leverage, even when reducing leverage would increase firm value. In contrast, the firm ex-post prefers to increase leverage further, which can decrease firm value. Admati and coauthors call this force the *leverage ratchet effect*,[1] arguing that it arises from an agency conflict between equity-holders and debt-holders. As increasing leverage makes the existing debt riskier, thereby reducing its market value, it may benefit equity-holders, even if such an increase ultimately destroys overall firm value. Conversely, equity holders stand to lose from leverage reductions, as the benefits accrue disproportionately to the bond-holders. Thus, the firm's equilibrium leverage rises above the optimal leverage predicted by static trade-off theory, leading to higher default costs. At the extreme, under the leverage policy described in DeMarzo and He (2021), the increment in expected default costs completely offsets the tax-shield benefits of issuing any debt.

Notably, the existing papers on the leverage ratchet effect restrict attention to straight debt, as the only instrument that is tax exempt. However, in practice many other forms of debt, which for instance include covenants or optionality, are routinely issued by firms and generate tax shields. In this paper, we ask whether other forms of debt can be used to mitigate the leverage ratchet effect without additional information requirements beyond those already assumed in the existing literature. We find that *demandable debt*, or demand deposits, are an effective solution to the leverage ratchet effect.[2]

In our analysis, we demonstrate that giving debt-holders the right to sell the debt back to the issuing firm at face value disciplines the firm and prevents excessive debt issuance. With demandable debt, the firm maintains a more conservative leverage ratio, which lowers default risk and benefits equity holders, as the tax shield advantages now outweigh the expected costs of default. Importantly, to enforce the demand clause one needs the exact same information structure required for straight debt, such as in Leland (1994). That is, enforcing the demand clause only requires: 1) that the leverage ratio is publicly observable, a standard assumption in the literature for debt pricing based on leverage; and 2) that bond-holders can demand the repayment of their principal invested at any time, triggering bankruptcy if the firm refuses.

We construct a dynamic model with discrete yet short time intervals to precisely capture the timing and information available for all firm actions. In our model, the firm operates a cash-generating asset that produces taxable cash flows influenced by normally distributed shocks and faces the classical trade-off problem of optimal capital structure. The firm can

[1]The concept of a leverage ratchet effect builds on earlier studies such as Black and Scholes (1973), Bizer and DeMarzo (1992), and Leland (1994).

[2]From now on, we are going to use demandable debt and demand deposits interchangeably.

issue tax-exempt demandable debt, but it then has the option to strategically default on any payment, incurring significant costs that, for simplicity, lead to the cessation of the asset's operations. The firm acts in the interest of its equity holders.

The intuition behind the optimality of demandable debt is as follows: as leverage rises, the debt's market price falls, due to the presence of default costs. Eventually, the price of debt will fall below its face value, prompting the debt-holders to exercise their demand clause and sell their debt back to the firm. Anticipating this response by the debt-holders, the firm is deterred from increasing its leverage above the target level. This mechanism results in a target debt ratio that always maintains the debt price at its face value, except when an extreme negative shocks precipitates the firm's default. This target ratio provides two main advantages to the firm. First, as the firm's asset base expands, it can issue additional debt to harness the tax shield benefits linked to asset growth. Second, by honoring the demand clause and adjusting its leverage following moderate cash-flow shocks, the firm reduces its leverage before these adverse shocks accumulate, thereby mitigating its risk of default.

The discreteness of time in our baseline model allows us to study the consequences of shortening the time intervals on firm value and default risk. We find that, as the time intervals shorten, the firm improves its capability to respond to shocks with near-instantaneous adjustments to its leverage level. This allows the firm to increase debt immediately after a positive cash flow shock, in order to capture nearly the full tax-shield benefits, and to promptly decrease leverage to minimize default risk after negative shocks. Asymptotically, as time becomes continuous, the firm achieves its full tax shield benefits without incurring default costs, reaching a valuation comparable to that of a fully committed, non-defaulting entity. Thus, we conclude that the leverage ratchet effect arises due to the restriction to straight debt, not due to the limited commitment friction per se.

Relation to the Literature. Our paper contributes to the literature on addressing the leverage ratchet effect. Starting from the seminal paper by DeMarzo and He (2021), both Qi (2018) and Malenko and Tsoy (2020) rely on non-Markov equilibria to resolve the leverage ratchet effect. Specifically, these papers use the equilibrium similar to that described in DeMarzo and He (2021) as a self-sustaining credible threat to punish any deviation from the target leverage policy. As a result, if the firm follows the target leverage policy, the threat is not used, allowing the firm to maintain a higher equity value. However, if the firm deviates from the target leverage policy, the self-sustaining credible threat is used, which severely decreases equity value. Rather than using an equilibrium that is highly history-dependent, our paper demonstrates that the leverage ratchet effect can be resolved within the framework of a classic Markov equilibrium.

Donaldson, Koont, Piacentino, and Vanasco (2024) explores how a credit line can prevent a firm from issuing additional debt. Their paper proposes that a credit line leads to what they term the "ratchet anti-ratchet effect," which effectively restricts the firm from leveraging further. Specifically, the lender anticipates that acquiring additional loans from the firm will incentivize the firm to draw on the credit line, which would significantly dilute the value of the additional loans. Consequently, the lender is unwilling to purchase these additional loans at a price acceptable to the firm, thereby preventing further leverage. However, while the credit line is effective in preventing leverage increases, it does not ensure that the firm can efficiently adjust its debt level—a critical capability when cash flow is volatile rather than constant, as assumed in their paper. In contrast, the demandable debt in our paper enables efficient debt adjustment. For instance, if the firm experiences negative cash flow, demandable debt incentivizes the firm to repurchase debt to mitigate excessive bankruptcy risk, whereas the credit line discussed in Donaldson et al. (2024) cannot provide this flexibility, and works in their environment only because the firm's cash flows are constant.

This paper also contributes to the literature on the adoption and implications of demandable debt. A significant focus in financial economics has been the role of demandable debt in providing liquidity insurance to investors when liquidity shocks are non-contractible. Diamond and Dybvig (1983) underscores demandable debt can effectively offer such insurance. Extending this discourse, Jacklin (1987) examines the conditions under which demandable debt outperforms dividend-paying securities in mitigating liquidity risks, emphasizing that the effectiveness hinges on the presence of trading restrictions. Further, Diamond and Rajan (2001) argues that demandable debt enables relationship lenders to commit to utilizing their specialized skills for collecting returns even after transferring holdings to other investors, thereby enhancing liquidity without sacrificing performance. Additionally, Calomiris and Kahn (1991) illustrates that demandable debt can attract deposits by giving creditors the option to force liquidation when there is a risk of bankers misappropriating funds. In such cases, debt-holders, upon receiving adverse signals, may exercise early withdrawal to safeguard their investments. Building on these insights, our paper introduces a novel function of demandable debt: aiding firms in sustaining an optimal capital structure and curbing excessive debt issuance. This mechanism offers a practical tool for aligning the interests of shareholders and debt-holders, contributing to financial stability by preventing over-leverage and mitigating default risks.

Our paper is also related to the literature on how trading frequency can enhance welfare. For example, Kreps (1982) and Duffie and Huang (1985) discuss how more frequent portfolio re-balancing allows investors to construct Arrow-Debreu securities with fewer assets. Our comparative static as time-intervals shorten shares a similar flavor: shorter period lengths and more frequent re-balancing imply that shocks in each period are likely smaller, allowing the firm to adjust its leverage ratio more promptly to capture the tax shield benefit and avoid default costs, resulting in a more efficient outcome. This contrasts with the standard intuition in commitment problems such as Coase (1972), where a shorter time period reduces the firm's commitment power, leading to greater welfare losses.

The paper is structured as follows. Section 2 presents the economic environment and the equilibrium concept. Section 3 describes the conjectured equilibrium strategies, pins down two necessary conditions of the equilibrium, and proves the existence of equilibrium. Section 4 discusses the implementation conditions and the welfare implication of using demandable debt. Section 5 concludes.

2 Model Setup

2.1 Economic Environment

We analyze a discrete-time model with periods indexed by $t = 0, 1, 2, \ldots$. All players are risk-neutral and discount future payoffs using the factor $\exp(-r\Delta)$ per period, where Δ represents the length of each time period.

The firm, acting in the interest of its shareholders, operates a cash-generating machine that produces $X_t\Delta$ at time $t \geq 1$. The cash flow level X_t follows a log-normal distribution, given by:

$$X_t = X_{t-1} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta} Z_t\right), \tag{1}$$

where $\mu < r$ is the percentage drift, σ is the percentage volatility, and Z_t is a standard normal random variable. The cash flow is subject to taxation at a rate of τ.

Additionally, the firm can issue or repurchase long-term debt at any date $t \geq 0$, which pays a coupon of $c\Delta$ each period and matures at a rate of $1 - \exp(-\xi\Delta)$ starting from the next period after issuance, to a competitive financial market. The coupon payments are tax-deductible up to α fraction of the cash flow. We assume that c and ξ are chosen so that the following condition holds:

Assumption 1. $c > \bar{c} := \frac{\exp(r\Delta)-1}{\Delta}$.

Violation of Assumption (1) implies that the present value of the debt remains below its face value if the firm has even a small chance of default, prompting all debt-holders to exercise the demand clause—discussed later—resulting in the trivial outcome where no debt remains outstanding when the debt is competitively priced. To see this, note that the discounted sum of all payments to a debt-holder with a unit face value is bounded by

$$\begin{aligned}
&\sum_{t\geq 1} \exp(-(r+\xi)\Delta)\,(c\Delta + 1 - \exp(-\xi\Delta)) \\
&= \exp(-r\Delta)\,(c\Delta + 1 - \exp(-\xi\Delta))\,\frac{1}{1-\exp\left(-(r+\xi)\Delta\right)},
\end{aligned}$$

which represents the present value of the debt assuming no default, and is less than 1 if Assumption (1) fails.

Let us denote the amount of newly issued or repurchased debt in period t as Γ_t, and the outstanding level of debt at the end of period t as F_t. Since the outstanding level of debt at the beginning of period t equals the outstanding level at the end of the previous period, we know that the outstanding debt at the beginning of period t is F_{t-1}.

The key distinction of this paper from the classical literature is the inclusion of a demand clause in the debt, which grants the debt-holder the option to demand full repayment of the principal at any time after the debt has been issued. Specifically, at any date t, after observing the realization of the cash flow level and the newly issued or repurchased amount of debt, the debt-holder who holds non-maturing debt can request the firm to early repay the face value of the debt in that period. If the firm honors the repayment, the debt matures. Following the literature, we assume that each debt-holder is atomless, making each one a price taker who cannot change the debt level by exercising the demand clause individually. However, the aggregation of the debt-holders' actions will change the debt level. We denote the amount of debt for which the debt-holders have exercised the demand clause at date t as D_t.

Given that debt matures at a rate $1 - \exp(-\xi\Delta)$, the firm issues or repurchases Γ_t, and the amount of debt with the demand clause exercised is denoted as D_t, the dynamics of the outstanding debt can be expressed as:

$$F_t = F_{t-1}\exp(-\xi\Delta) + \Gamma_t - D_t. \tag{2}$$

The firm also has the strategic option not to honor its debt payments to the debt-holders. To be more specific, in our paper, the firm may refuse to pay the coupon and maturing debt, and may also refuse to pay the debt for which the demand clause is exercised. In either case, the firm declares bankruptcy and incurs significant bankruptcy costs. For simplicity, we assume that bankruptcy costs equal the continuation value of the firm going forward. That is, the firm forfeits all future cash flows after it declares bankruptcy.

To summarize, we can decompose any period t into the following four stages:

1. The current cash flow level X_t is realized, and the firm collects $X_t\Delta$ as pre-tax profit. The firm must decide whether to honor its various liabilities related to taxes, coupons, and maturing debt, denoted by the indicator function $\mathbb{1}_t^{b1}$.

   (a) If the firm decides to honor its obligations:

   - The firm pays the coupon $CF_{t-1}\Delta$,
   - The firm pays the maturing debt $(1 - \exp(-\xi\Delta))F_{t-1}$,
   - The firm pays the tax $\tau X_t\Delta - \tau \min(\alpha X_t, CF_{t-1})\Delta$.

   After these payments, the game moves to the next stage.

   (b) If the firm decides not to honor its obligations:

   - The firm declares bankruptcy.
   - In this case, the game ends with zero continuation value for all players.

2. The firm repurchases or issues additional debt Γ_t.

3. Debt-holders (including those who just purchased the debt in this period) decide whether to exercise the demand clause, which we denote as $\mathbb{1}_t^d$. The aggregate level of debt with demand clause exercised is denoted as D_t.

4. The firm decides whether to honor its promise on the demand clause, which we denote as $\mathbb{1}_t^{b2}$.

   (a) If the firm decides to honor, the firm pays D_t. The game continues to period $t+1$, the cash flow level follows (1), and the debt level follows (2).

   (b) If the firm decides not to honor, the game ends and players receive 0.

2.2 Equilibrium Concept

2.2.1 Strategy

In this paper, we focus on the Markov Perfect Equilibrium (MPE). At any date t, the firm decides whether to pay coupon and maturing debt based on the current cash flow level X_t and the debt level at the end of the last period F_{t-1}. Formally, it has a default policy $\mathbb{1}^{b1}$, so that its default choice $\mathbb{1}_t^{b1} = \mathbb{1}^{b1}(X_t, F_{t-1})$ is a random variable over $\{0, 1\}$, where 0 signifies not honoring the payment, and 1 signifies honoring the payment.

If the firm decides to pay the coupon and maturing debt, it issues or repurchases debt using the policy Γ. Similarly, this decision is based on the current cash flow level X_t and the previous period's debt F_{t-1}, so $\Gamma_t := \Gamma(X_t, F_{t-1})$ is a random variable over $[-\exp(-\xi\Delta)F_{t-1}, \infty)$, where the lower bound ensures that the firm cannot repurchase more debt than the outstanding amount of debt which is not maturing.

After observing the realization of Γ_t[3], together with X_t and F_{t-1}, the debt-holders decide whether to exercise the demand clause at date t. Formally, $\mathbb{1}_t^d = \mathbb{1}^d(X_t, F_{t-1}, \Gamma_t)$ is a random variable over $\{0, 1\}$, where 0 signifies not exercising the demand clause, and 1 signifies exercising it.

Given that each debt-holder is atomless and the law of large number, D_t is determined by the aggregate actions of all debt-holders according to the following equation:

$$D_t := D(X_t, F_{t-1}, \Gamma_t) = (\exp(-\xi\Delta)F_{t-1} + \Gamma_t)\mathbb{E}[\mathbb{1}_t^d \mid X_t, F_{t-1}, \Gamma_t], \tag{3}$$

where the term $\exp(-\xi\Delta)F_{t-1}+\Gamma_t$ represents the amount of debt the firm has after paying the maturing debt and issuing or repurchasing additional debt, and $\mathbb{E}[\mathbb{1}_t^d \mid X_t, F_{t-1}, \Gamma_t]$ represents the expected fraction of debt-holders who exercise the demand clause at time t.

Subsequently, the firm decides whether to repay D_t at date t, based on X_t, F_{t-1}, Γ_t, and D_t. Formally, $\mathbb{1}_t^{b2} := \mathbb{1}^{b2}(X_t, F_{t-1}, \Gamma_t, D_t)$ is a random variable over $\{0, 1\}$, where 0 represents not honoring the payment, and 1 represents honoring the payment.

2.2.2 Payoff

Given this Markovian structure, we can express the payoffs for equity-holders and debt-holders recursively. To be more specific, at the beginning of period t, we can denote the equity value as $V_e^-(X_t, F_{t-1})$, and market-to-par ratio of debt as $V_d^-(X_t, F_{t-1})$. Similarly, at the end of period t, conditional on the firm not declaring bankrupcty in any stage, the equity value at the end of date t as $V_e^+(X_t, F_t)$ and the market-to-par-ratio of debt as $V_d^+(X_t, F_t)$.

Throughout this paper, we assume that the debt is competitively priced by the market. As a result, the debt price at the end of date t depends only on X_t and F_t, conditional on the firm not declaring bankruptcy. We denote this debt price as $P_t := P(X_t, F_t) = V_d^+(X_t, F_t)$.

Given initial states (X_t, F_{t-1}), the strategies $\left(\mathbb{1}^{1b}, \Gamma, \mathbb{1}^d, \mathbb{1}^{2b}\right)$, and the price policy P, let us consider any possible deviation $\left(\mathbb{1}_t^{b1'}, \Gamma_t', D_t', \mathbb{1}_t^{b2'}\right)$ at date t.

When the firm issues/repurchases Γ_t' amount of debt, the price at which the debt is issued/repurchased is calculated based on the debt-holder's strategy of exercising the demand clause and the firm's strategy of whether to honor the demand clause. To be more specific, by competitive pricing, the price level of debt at which the debt is issued/repurchased at date t is:

[3]With a little abuse of notation, let us use Γ_t to denote the realization as well.

$$\tilde{P}'_t = \mathbb{E}\left\{\mathbb{1}^{b2}_t(X_t, F_{t-1}, \Gamma'_t, D_t)P(X_t, \tilde{F}'_t)|X_t, F_{t-1}, \Gamma'_t\right\},$$

where

$$D_t = \left(F_{t-1}\exp(-\xi\Delta) + \Gamma'_t\right)\mathbb{E}\left[\mathbb{1}^d(X_t, F_{t-1}, \Gamma'_t)|X_t, F_{t-1}, \Gamma'_t\right],$$

and

$$\tilde{F}'_t = F_{t-1}\exp(-\xi\Delta) + \Gamma'_t - D_t.$$

In addition, given Γ'_t and the potential deviation of debt-holders D'_t, the debt level at the end of date t, can be written as:

$$F'_t = F_{t-1}\exp(-\xi\Delta) + \Gamma'_t - D'_t.$$

As a result, the equity value at the beginning of date t, with possible actions $\left(\mathbb{1}^{b1'}_t, \Gamma'_t, D'_t, \mathbb{1}^{b2'}_t\right)$, can be written as

$$\begin{aligned} V^-_e\left(\mathbb{1}^{b1'}_t, \Gamma'_t, D'_t, \mathbb{1}^{b2'}_t\right) = X_t\Delta + \mathbb{1}^{b1'}_t\Big[&\left(-\tau X_t + \tau\min\{\alpha X_t, cF_{t-1}\} - cF_{t-1}\right)\Delta \\ - \left(1 - \exp(-\xi\Delta)\right)F_{t-1} + \tilde{P}'_t\Gamma'_t + \mathbb{1}^{b2'}_t&\left(-D'_t + V^+_e(X_t, F'_t)\right)\Big]. \end{aligned} \tag{4}$$

For any debt with the demand clause exercise action $\mathbb{1}^{d'}_t$, the market-to-par-ratio of debt at the beginning of date t can be written as

$$\begin{aligned} V^-_d\left(\mathbb{1}^{b1'}_t, \Gamma'_t, D'_t, \mathbb{1}^{b2'}_t|\mathbb{1}^{d'}_t\right) = \mathbb{1}^{b1'}_t\Big[&c\Delta + 1 - \exp(-\xi\Delta) \\ &+ \mathbb{1}^{b2'}_t\exp(-\xi\Delta)\left(\mathbb{1}^{d'}_t + (1 - \mathbb{1}^{d'}_t)V^+_d(X_t, F'_t)\right)\Big]. \end{aligned} \tag{5}$$

By definition, the equity value and the debt value at the beginning of period t, $V^-_e(V_t, F_{t-1})$ and $V^-_d(V_t, F_{t-1})$, can be calculated from equations (4) and (5) with $\mathbb{1}^{b1}_t$, Γ_t, D_t, and $\mathbb{1}^{b2}_t$ following strategies $\left(\mathbb{1}^{1b}, \Gamma, \mathbb{1}^d, \mathbb{1}^{2b}\right)$:

$$V^-_e(X_t, F_{t-1}) = \mathbb{E}\left\{V^-_e\left(\mathbb{1}^{b1}_t, \Gamma_t, D_t, \mathbb{1}^{b2}_t\right)|X_t, F_{t-1}\right\}, \tag{6}$$

and

$$V^-_d(X_t, F_{t-1}) = \mathbb{E}\left\{V^-_d(\mathbb{1}^{b1}_t, \Gamma_t, D_t, \mathbb{1}^{b2}_t)|X_t, F_{t-1}\right\}. \tag{7}$$

Moreover, we have the following recursive equations between the values at the end of the period and the values at the beginning of the next period:

$$V_e^+(X_t, F_t) = \mathbb{E}\{\exp(-r\Delta)V_e^-(X_{t+1}, F_t)|X_t, F_t\}, \tag{8}$$

and

$$V_d^+(X_t, F_t) = \mathbb{E}\{\exp(-r\Delta)V_d^-(X_{t+1}, F_t)|X_t, F_t\}. \tag{9}$$

2.2.3 Equilibrium Conditions

Let $\mathbb{1}_t^{b1}$, Γ_t, D_t, $\mathbb{1}_t^d$, and $\mathbb{1}_t^{b2}$ follow strategies $\left(\mathbb{1}^{1b}, \Gamma, \mathbb{1}^d, \mathbb{1}^{2b}\right)$. By one-shot deviation principle, the strategies $\left(\mathbb{1}^{1b}, \Gamma, \mathbb{1}^d, \mathbb{1}^{2b}\right)$ construct an equilibrium if and only if

1. **Optimality of Equity Holder**

   - (Optimality of Default Policy)

     – Given any (X_t, F_{t-1}) and possible deviation $\mathbb{1}_t^{b1'}$, we have

     $$V_e^-(X_t, F_{t-1}) \geq \mathbb{E}\left\{V_e^-\left(\mathbb{1}_t^{b1'}, \Gamma_t, D_t, \mathbb{1}_t^{b2}\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}\right\} \tag{10}$$

     – Given any $\left(X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma'_t, D'_t\right)$ and possible deviation $\mathbb{1}_t^{b2'}$, we have

     $$\begin{aligned}&\mathbb{E}\left\{V_e^-\left(\mathbb{1}_t^{b1'}, \Gamma'_t, D'_t, \mathbb{1}_t^{b2}\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma'_t, D'_t\right\}\\ &\geq \mathbb{E}\left\{V_e^-\left(\mathbb{1}_t^{b1'}, \Gamma'_t, D'_t, \mathbb{1}_t^{b2'}\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma'_t, D'_t, \mathbb{1}_t^{b2'}\right\}\end{aligned} \tag{11}$$

   - (Optimality of Debt Policy) Given any $\left(X_t, F_{t-1}, \mathbb{1}_t^{b1'}\right)$ and possible deviation Γ'_t , we have

     $$\begin{aligned}&\mathbb{E}\left\{V_e^-\left(\mathbb{1}_t^{b1'}, \Gamma_t, D_t, \mathbb{1}_t^{b2}\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}\right\}\\ &\geq \mathbb{E}\left\{V_e^-\left(\mathbb{1}_t^{b1'}, \Gamma'_t, D_t, \mathbb{1}_t^{b2}\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma'_t\right\}\end{aligned} \tag{12}$$

2. **Optimality of Debt Holder** Given any $(X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma_t')$,

$$\mathbb{E}\left\{V_d^-\left(\mathbb{1}_t^{b1'}, \Gamma_t', D_t, \mathbb{1}_t^{b2}\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma_t'\right\}$$
$$= \max\left\{\mathbb{E}\left\{V_d^-\left(\mathbb{1}_t^{b1'}, \Gamma_t', D_t, \mathbb{1}_t^{b2}|\mathbb{1}_t^{d'} = 0\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma_t'\right\},\right. \tag{13}$$
$$\left.\mathbb{E}\left\{V_d^-\left(\mathbb{1}_t^{b1'}, \Gamma_t', D_t, \mathbb{1}_t^{b2}\middle| \mathbb{1}_t^{d'} = 1\right)|X_t, F_{t-1}, \mathbb{1}_t^{b1'}, \Gamma_t'\right\}\right\}$$

3. **Competitive Pricing of Debt** $P(X_t, F_t) = V_d^+(X_t, F_t)$.

Equation 10 reflects that the firm optimally decides whether to honor its various liabilities related to taxes, coupons, and maturing debt. Intuitively, the firm will honor these payments if and only if the continuation value of the equity holders exceeds the amount of these payments in equilibrium. Similarly, Equation 11 shows that the firm optimally decides whether to honor the demand clause. In equilibrium, the firm will do so if and only if the continuation value exceeds the repayment amount of the debt on which the demand clause is exercised. Notably, if the firm follows its equilibrium debt issuance policy and the debt-holders follow their equilibrium demand clause exercise policy, the firm will honor the demand clause payment in Equation11, provided it is optimal for the firm to honor the various liabilities in Equation 10. This is because if the firm defaults on the demand clause, it forfeits future income and continuation value, making it incentive incompatible to fulfill various liabilities in the beginning. Equation 12 describes the firm's optimal decision regarding debt issuance or repurchase, aimed at maximizing the equity value. Finally, Equation 11 reflects the debt-holders' optimal exercise of the demand clause. Intuitively, they will refrain from exercising the demand clause if the market price of debt is higher than the face value, and will exercise the clause if the market price is lower than the face value.

3 Equilibrium

3.1 Conjectured Equilibrium

In this paper, we show that as $\Delta \to 0$, there exists an equilibrium that depends crucially on the debt ratio $f_t^- = \frac{F_{t-1}}{X_t}$ and $f_t^+ = \frac{F_t}{X_t}$ at any time t, given a properly designed coupon rate c. More specifically, there is a threshold $\bar{f}$ such that the firm chooses to honor its various liabilities at date t if and only if $\frac{F_{t-1}}{X_t} < \bar{f}$, and adjusts its debt level to maintain a target debt ratio of $\frac{\alpha}{c}$ if it chooses to honor— that is, after repaying the maturing debt, the firm issues or repurchases debt, choosing Γ_t such that the post-adjustment debt ratio

$$\tilde{f}_t := \frac{\Gamma_t}{X_t} + \exp(-\xi\Delta) f_t^- = \frac{\alpha}{c}.$$

The debt-holders, in turn, use their demand clause as a disciplinary tool to ensure that the firm adheres to this equilibrium strategy. We demonstrate that there exists a coupon rate c such that debt-holders will exercise their demand clause if $\tilde{f}_t > \frac{\alpha}{c}$, thereby forcing the firm to reduce f_t^+ to the target level. On the other hand, if $\tilde{f}_t \leq \frac{\alpha}{c}$, debt-holders will not exercise the demand clause.

Normalized Values

Given the conjecture that what matters for the equilibrium strategies and values are the debt ratio f_t^- and the debt ratio f_t^+, the cash flow level X_t serves only as a multiplier for equity values[4]. As a result, we can define $V_e^-(f_t^-)$, $V_d^-(f_t^-)$, $V_e^+(f_t^+)$, and $V_d^+(f_t^+)$ to represent the equity values per unit of cash flow and the debt value as functions of debt ratios as follows:

$$V_e^-(f_t^-) = \frac{V_e^-(X_t, F_{t-1})}{X_t}, \quad V_d^-(f_t^-) = V_d^-(X_t, F_{t-1}),$$

$$V_e^+(f_t^+) = \frac{V_e^+(X_t, F_t)}{X_t}, \quad V_d^+(f_t^+) = V_d^+(X_t, F_t).$$

Similarly, we can normalize the amount of debt issuance/repurchase and the amount of debt for which the demand clause has been exercised with respect to X_t, the firm's current cash flow. Specifically, we define:

$$\gamma(f_t^-) = \frac{\Gamma(X_t, F_{t-1})}{X_t}, \quad d(f_t^-, \gamma_t) = \frac{D(X_t, F_{t-1}, \gamma_t X_t)}{X_t}.$$

With some abuse of notation, we represent other strategies as functions of debt ratios as follows:

$$\mathbb{1}^{b1}(f_t^-) = \frac{\mathbb{1}^{b1}(X_t, F_{t-1})}{X_t}, \quad \mathbb{1}^{d}(f_t^-, \gamma_t) = \frac{\mathbb{1}^{d}(X_t, F_{t-1}, \gamma_t X_t)}{X_t},$$

$$\mathbb{1}^{b2}(f_t^-, \gamma_t, d_t) = \frac{\mathbb{1}^{b2}(X_t, F_{t-1}, \gamma_t X_t, d_t X_t)}{X_t}, \quad P(f_t^+) = \frac{P(X_t, F_t)}{X_t} = V_d^+(f_t^+).$$

We can now formalize the conjectured equilibrium as follows:

1. **Firm's Strategies**

[4] Notice that we already normalize the debt value by defining it as the debt value per face value.

(a) **Strategy for Honoring Various Liabilities:**

$$\mathbb{1}^{b1}(f_t^-) = \begin{cases} 1, & \text{if } f_t^- < \bar{f} \\ 0, & \text{if } f_t^- \geq \bar{f} \end{cases}$$

This implies that the firm will honor its various liabilities if and only if its debt ratio is below the threshold $\bar{f}$.

(b) **Strategy for Debt Issuance/Repurchase:**

$$\gamma(f_t^-) = \begin{cases} \frac{\alpha}{c} - \exp(-\xi\Delta) f_t^-, & \text{if } \exp(-\xi\Delta) f_t^- - \frac{\alpha}{c} < V_e^+\left(\frac{\alpha}{c}\right) \\ 0, & \text{if } \exp(-\xi\Delta) f_t^- - \frac{\alpha}{c} \geq V_e^+\left(\frac{\alpha}{c}\right) \end{cases}$$

Here, the firm issues or repurchases debt to maintain the debt ratio $\frac{\alpha}{c}$, unless the cost of doing so exceeds the continuation value of equity.

(c) **Strategy for Honoring the Demand Clause:**

$$\mathbb{1}^{b2}(f_t^-, \gamma_t, d_t) = \begin{cases} 1, & \text{if } d_t < V_e^+\left(\frac{\alpha}{c}\right) \\ 0, & \text{if } d_t \geq V_e^+\left(\frac{\alpha}{c}\right) \end{cases}$$

This strategy indicates that the firm will honor the payment related to the demand clause if the aggregate amount of exercised demand clause is smaller than the continuation value of equity.

2. **Debt-Holders' Strategy:**

$$\mathbb{1}^{d}(f_t^-, \gamma_t, d_t) = \begin{cases} \begin{cases} 1 & \text{with probability } \frac{\tilde{f}_t - \frac{\alpha}{c}}{\tilde{f}_t} \\ 0 & \text{with probability } \frac{\frac{\alpha}{c}}{\tilde{f}_t} \end{cases}, & \text{if } \tilde{f}_t > \frac{\alpha}{c} \\ 0, & \text{if } \tilde{f}_t \leq \frac{\alpha}{c} \end{cases}$$

Here, debt-holders use their demand clause as a disciplinary tool, exercising it with a probability proportional to the excess debt ratio above the target $\frac{\alpha}{c}$. If the debt ratio is at or below the target, they do not exercise the demand clause.

$1a$ of firm's strategies follows directly from the definition of the threshold $\bar{f}$, which governs the decision to honor various liabilities. The first case in $1b$ of firm's strategies arises from the conjecture that the firm targets a debt ratio of $\frac{\alpha}{c}$. Debt-holders' strategy is based on the idea that they exercise the demand clause to correct any deviation from the firm's equilibrium strategy. All other strategies are designed as some off-path actions, and we will show later that those conjectured strategies indeed constitute an equilibrium.

3.2 Necessary Conditions

To establish our conjecture as an equilibrium, we are going to show two necessary conditions. One necessary condition reflects Equation (13) to ensure the optimality of the debt-holders' strategy. The other necessary condition reflects Equation (10) to ensure that (1a) in the firm's strategies is optimal.

Necessary Condition for Optimal Demand Clause Exercise

Before we formally derive the necessary condition, it is useful to first describe the subgame equilibrium at period $t+1$ given f_t^+ at the end of date t.

Subgame Equilibrium at Period $t+1$: Given the conjecture of (1a) in the firm's strategies, we know that the firm will honor its various liabilities at date $t+1$ if and only if $f_{t+1}^- < \bar{f}$. As we have argued earlier, if the firm honors its various liabilities, it must honor the following demand clause payment. Otherwise, the firm would pay various liabilities but receive zero continuation value, which implies it is not incentive compatible. If $\exp(-\xi\Delta)f_{t+1}^- - \frac{\alpha}{c} \geq V_e^+\left(\frac{\alpha}{c}\right)$, (1b) in the firm's strategies, debt-holders' strategy, and (1c) of the firm's strategies imply that the firm has too much demand clause to repurchase, making honoring the demand clause no longer incentive compatible. As a result, as long as the firm honors its various liabilities at date $t+1$, we must have $\exp(-\xi\Delta)f_{t+1}^- - \frac{\alpha}{c} < V_e^+\left(\frac{\alpha}{c}\right)$, and the firm will issue/repurchase debt so that $\tilde{f}_{t+1} = \frac{\alpha}{c}$ at period $t+1$ according to (1b) in the firm's strategies.

Moreover, let us define

$$Z(f_t^+) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{f_t^+}{\bar{f}} - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right).$$

Given Equation (1), the dynamics of the cash flow, we have $f_{t+1}^- < \bar{f}$ if and only if the cash flow does not experience a large negative shock. That is, $Z_{t+1} > Z(f_t^+)$. As a result, $f_{t+1}^- < \bar{f}$ occurs with probability $1 - \Phi\left(Z(f_t^+)\right)$, where $\Phi(\cdot)$ represents the normal distribution.

Deriving Necessary Condition: By our conjecture that the debt-holders will exercise the demand clause to maintain the debt ratio $f_t^+ = \frac{\alpha}{c}$ if possible, and the equilibrium condition (10), the debt-holders should be indifferent to exercising the demand clause or not, given the market price $P\left(\frac{\alpha}{c}\right)$, which implies that $P\left(\frac{\alpha}{c}\right)$ equals its face value of 1.

Given any f_t^+ at the end of period t, if the firm experiences a large negative cash flow shock, $Z(f_t) \leq Z_{t+1}$, the firm will default, and the debt-holder will receive nothing; otherwise, the firm will honor its various liabilities, and demand clause. In the latter case, the debt-holder with a unit of face value will receive coupon $c\Delta$, have $1-\exp(-\xi\Delta)$ debt matured, and receive $P\left(\frac{\alpha}{c}\right)=1$ as the price for the remaining $\exp(-\xi\Delta)$ debt. In total, this gives the debt-holder $c\Delta+1$. Given the probability of the cash flow shock, the price of debt at the end of period t is

$$P(f_t^+)=\exp(-r\Delta)\left(1-\Phi\left(Z(f_t^+)\right)\right)(c\Delta+1). \tag{14}$$

Notice that $\bar{f}$ and c are two important parameters determining the debt price. $\bar{f}$, which is the bankruptcy threshold, determines how likely the firm is going to receive the next period's payments through $Z\left(\frac{\alpha}{c}\right)$, and c, which is the coupon rate, determines the amount of payment, given there is no default. Let us denote $Q_1(\bar{f},c)=P\left(\frac{\alpha}{c}\right)$ to explicitly express how $\bar{f}$ and c impact the debt price when the firm has $\frac{\alpha}{c}$ debt outstanding at the end of the period. Given $P\left(\frac{\alpha}{c}\right)=1$, $\bar{f}$ and c must necessarily follow the following relationship:

$$Q_1(\bar{f},c)=\exp(-r\Delta)\left(1-\Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right)(c\Delta+1)=1. \tag{15}$$

Necessary Condition for Honoring Various Liabilities

If the firm chooses to honor its various liabilities at period t, its continuation value depends on $\exp(-\xi\Delta)f_t^-$. If $\exp(-\xi\Delta)f_t^-<V_e^+\left(\frac{\alpha}{c}\right)+\frac{\alpha}{c}$, the firm will issue/repurchase $\frac{\alpha}{c}-\exp(-\xi\Delta)f_t^-$ amount of debt, no debt-holders will exercise the demand clause, and the firm will trivially honor the demand clause according to our conjectured equilibrium strategy. Since the debt price at the end of the period is $P\left(\frac{\alpha}{c}\right)=1$ and the firm will honor the demand clause at this period, the price $\tilde{P}_t'$ at which the debt will be issued/repurchased also equals 1. In this case, the continuation value of the firm after honoring its various liabilities at period t is $\frac{\alpha}{c}-\exp(-\xi\Delta)f_t^-+V_e^+\left(\frac{\alpha}{c}\right)$. If $\exp(-\xi\Delta)f_t^-\geq V_e^+\left(\frac{\alpha}{c}\right)+\frac{\alpha}{c}$, the firm will not issue/repurchase any debt and subsequently default on the demand clause according to the conjectured equilibrium strategy. In this case, the continuation value of the firm is 0. As a result, the continuation value of the firm by honoring its various liabilities is $\max\{\frac{\alpha}{c}-\exp(-\xi\Delta)f_t^-+V_e^+\left(\frac{\alpha}{c}\right),0\}$. On the other hand, in order to honor its various liabilities, the firm pays $\tau\Delta-\tau\min\{\alpha,cf_t^-\}\Delta$ in tax, $cf_t^-\Delta$ in coupon payments, and $\left(1-\exp(-\xi\Delta)\right)f_t^-$ for maturing debt.

According to (1a) of the firm's strategies and Equation (10), the firm should be indifferent between honoring its various liabilities or not when $f_t^-=\bar{f}$. Therefore, we have the following necessary condition:

$$\begin{aligned}&\left(-\tau+\tau\min\{\alpha,c\bar{f}\}-c\bar{f}\right)\Delta-\left(1-\exp(-\xi\Delta)\right)\bar{f}\\&+\max\left\{\frac{\alpha}{c}-\exp(-\xi\Delta)\bar{f}+V_e^+\left(\frac{\alpha}{c}\right),0\right\}=0.\end{aligned}$$

We can rewrite the left-hand side of the above equation as $Q_2(\bar{f}, c)$, to emphasize that it crucially depends on both $\bar{f}$ and c. The necessary condition can then be restated as follows:

$$Q_2(\bar{f}, c) := \left(-\tau + \tau \min\{\alpha, c\bar{f}\} - c\bar{f}\right)\Delta + \frac{\alpha}{c} - \bar{f} + V_e^+\left(\frac{\alpha}{c}\right) = 0. \tag{16}$$

Determination of $V_e^+(\cdot)$: Notice that Equation (16) crucially depends on $V_e^+\left(\frac{\alpha}{c}\right)$, which is endogenously determined in equilibrium. Indeed, $V_e^+\left(\frac{\alpha}{c}\right)$ can be derived from the recursive form of equity values.

Specifically, by Equations (4) and (8), for any f_t^+, we have

$$\begin{aligned}
&V_e^+(f_t^+) = \exp\left(-r\Delta\right)\Delta \int_{-\infty}^{\infty} g(Z_{t+1})\phi(Z_{t+1})dZ_{t+1} \\
&+ \exp\left(-r\Delta\right)\Delta \int_{Z(f_t^+)}^{\infty} \left(-\tau g(Z_{t+1}) + \tau \min\{cf_t^+, \alpha g(Z_{t+1})\} - cf_t^+\right)\phi(Z_{t+1})dZ_{t+1} \\
&+ \exp\left(-r\Delta\right)\left[-f_t^+ \int_{Z(f_t^+)}^{\infty} \phi(Z_{t+1})dZ_{t+1} + \left(\frac{\alpha}{c} + V_e\left(\frac{\alpha}{c}\right)\right)\int_{Z(f_t^+)}^{\infty} g(Z_{t+1})\phi(Z_{t+1})dZ_{t+1}\right],
\end{aligned} \tag{17}$$

where $\phi(\cdot)$ represents the density function of a standard normal distribution. To interpret the equation, the first line represents the firm's pre-tax income for the next period, the second line represents the firm's tax and coupon payments, and the third line represents the firm's payoff from debt level adjustment and its future continuation value.

Let us define

$$Z_c(f_t^+) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{c}{\alpha}f_t^+ - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right)$$

to represent the cash flow shock such that $cf_t^+ = \alpha g(Z_{t+1})$, and

$$\tilde{Z}(f_t^+) = \max\{Z_c(f_t^+), Z(f_t^+)\}$$

to better capture the tax shield calculation.

Therefore, we know that the firm is going to default if it experiences a negative cash flow $Z_{t+1} \leq Z(f_t^+)$, the firm is going to collect a tax shield $\alpha g(Z_{t+1})$ if it experiences a cash flow $Z_{t+1} \in (Z(f_t^+), \tilde{Z}(f_t^+)]$, and the firm is going to collect a tax shield τf_t^+ if the firm has a cash flow $Z_{t+1} > \tilde{Z}(f_t^+)$. By properties of the normal distribution, we have

$$\begin{aligned} V_e^+(f_t^+) = \exp(-r\Delta) \Bigg[& \Delta \exp(\mu\Delta) - \tau\Delta \exp(\mu\Delta) \left[1 - \Phi\left(Z(f_t^+) - \sigma\sqrt{\Delta}\right)\right] \\ & + \tau\alpha\Delta \exp(\mu\Delta) \left[\Phi\left(\tilde{Z}(f_t^+) - \sigma\sqrt{\Delta}\right) - \Phi\left(Z(f_t^+) - \sigma\sqrt{\Delta}\right)\right] \\ & + \tau c f_t^+ \Delta \left[1 - \Phi\left(\tilde{Z}(f_t^+)\right)\right] - \left(c f_t^+ \Delta + f_t^+\right) \left[1 - \Phi\left(Z(f_t^+)\right)\right] \\ & + \left[\frac{\alpha}{c} + V_e^+\left(\frac{\alpha}{c}\right)\right] \exp(\mu\Delta) \left[1 - \Phi\left(Z(f_t^+) - \sigma\sqrt{\Delta}\right)\right] \Bigg]. \end{aligned} \tag{18}$$

To interpret each term in Equation (18), $\Delta \exp(\mu\Delta)$ represents the expected cash flow next period,

$$-\tau\Delta \exp(\mu\Delta) \left[1 - \Phi\left(Z(f_t^+) - \sigma\sqrt{\Delta}\right)\right]$$

represents the expected tax payment (without tax shield) next period,

$$\tau\alpha\Delta \exp(\mu\Delta) \left[\Phi\left(\tilde{Z}(f_t^+) - \sigma\sqrt{\Delta}\right) - \Phi\left(Z(f_t^+) - \sigma\sqrt{\Delta}\right)\right] + \tau c f_t^+ \Delta \left[1 - \Phi\left(\tilde{Z}(f_t^+)\right)\right]$$

represents the expected tax shield next period,

$$-c f_t^+ \Delta \left[1 - \Phi\left(Z(f_t^+)\right)\right]$$

represents the expected coupon payment next period,

$$-f_t^+ \left[1 - \Phi\left(Z(f_t^+)\right)\right] + \frac{\alpha}{c} \exp(\mu\Delta) \left[1 - \Phi\left(Z(f_t^+) - \sigma\sqrt{\Delta}\right)\right]$$

represents the expected payoff from debt level change (including debt maturing, repurchase, and issuance) next period, and

$$V_e^+\left(\frac{\alpha}{c}\right) \exp(\mu\Delta) \left[1 - \Phi\left(Z(f_t^+) - \sigma\sqrt{\Delta}\right)\right]$$

represents the expected continuation value from future operation after next period.

Plugging $f_t^+ = \frac{\alpha}{c}$ into Equation (18), we can derive $V_e^+\left(\frac{\alpha}{c}\right)$:

$$
\begin{aligned}
V_e^+\left(\frac{\alpha}{c}\right) =& \frac{\exp(-r\Delta)}{1-\exp\left(-(r-\mu)\Delta\right)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]} \\
&\Bigg[\Delta\exp(\mu\Delta) - \tau\Delta\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right] \\
&+\tau\alpha\Delta\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right] \\
&+\tau\alpha\Delta\left[1-\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)\right)\right]-\left(\alpha\Delta+\frac{\alpha}{c}\right)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right] \\
&+\frac{\alpha}{c}\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]\Bigg].
\end{aligned}
\tag{19}
$$

In addition, we derive the value of $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right)$, $\lim_{\bar{f}\to 0} V_e^+\left(\frac{\alpha}{c}\right)$, and $\lim_{\bar{f}\to\infty} V_e^+\left(\frac{\alpha}{c}\right)$ in Lemma 1. These results will be useful in the following analysis. To simplify the expressions, we define the following notations:

$$
V_e^{+\Delta\to 0}(c) := \frac{1}{r-\mu}\left(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{c}\mu\right),
$$

$$
V_e^{+\bar{f}\to 0}(c) := \exp(-(r-\mu)\Delta)\Delta,
$$

and

$$
\begin{aligned}
V_e^{+\bar{f}\to\infty}(c) := \frac{\exp(-r\Delta)}{1-\exp(-(r-\mu)\Delta)}&\Bigg[(1-\tau)\Delta\exp(\mu\Delta)+\tau\alpha\Delta\exp(\mu\Delta)\left[\Phi\left(-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)\sqrt{\Delta}\right)\right] \\
&+\tau\alpha\Delta\left[1-\Phi\left(-\left(\frac{\mu}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\Delta}\right)\right]-\alpha\Delta+\frac{\alpha}{c}\left[\exp(\mu\Delta)-1\right]\Bigg].
\end{aligned}
$$

Lemma 1. *For any $\bar{f} > \frac{\alpha}{c}$, $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right) = V_e^{+\Delta\to 0}(c)$. In addition, $\lim_{\bar{f}\to 0} V_e^+\left(\frac{\alpha}{c}\right) = V_e^{+\bar{f}\to 0}(c)$, which is positive, and $\lim_{\bar{f}\to\infty} V_e^+\left(\frac{\alpha}{c}\right) = V_e^{+\bar{f}\to\infty}(c)$, which is bounded.*

Meeting Necessary Conditions

Now, we need to establish that there exist $(\bar{f}, c)$ such that two necessary conditions, Equations (15) and (16), hold.

First, we show that for any c satisfying Assumption 1, there exists a function $\bar{f}^{Q_1}(c)$ such that $(\bar{f}^{Q_1}(c), c)$ satisfies Equation (15). The intuition is that, given Assumption 1, the firm receives a coupon payment larger than the discounting factor, making the present value of the debt strictly greater than the face value in the absence of default risk. In this scenario, if the default boundary $\bar{f}$ is very high, the firm is highly unlikely to default, approximating a no-default situation and resulting in a higher expected present value than the face value. Conversely, if the default boundary $\bar{f}$ is very low, the firm is highly likely to default, rendering the debt nearly worthless and leading to an expected present value significantly below the face value. To satisfy Equation (15), there exists an intermediate $\bar{f}$ representing a moderate default risk, ensuring that the expected present value precisely equals the face value. Given the continuity of $Q_1(\bar{f}, c)$, $\bar{f}^{Q_1}(c)$ can be constructed as continuous.

Second, when Δ is not large, we show that for any c, there exists a function $\bar{f}^{Q_2}(c)$ such that $(\bar{f}^{Q_2}(c), c)$ satisfies Equation (16). In this case, when the debt level is low, the firm has a manageable debt burden, resulting in a continuation value that exceeds its liabilities, including both taxes and debt, thus dissuading default. Conversely, when the debt level is high, the firm faces a significant debt burden, which incentivizes default. Consequently, there exists a medium level of debt burden $\bar{f}^{Q_2}(c)$ at which the firm is indifferent between defaulting and continuing operations. Given the continuity of $Q_2(\bar{f}, c)$, $\bar{f}^{Q_2}(c)$ can also be constructed as continuous.

Therefore, as long as we can have a $c > \bar{c}$ such that $\bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$, this c and $\bar{f} := \bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$ will satisfy both necessary conditions. We show that such a c exists when Δ is close to zero. The intuition is as follows:

First, when $\Delta \to 0$, $\bar{f}^{Q_2}(c) \to \frac{\alpha}{c} + V_e^{+\Delta\to 0}(c)$. This is because the firm needs to retire $\bar{f}^{Q_2}(c) - \frac{\alpha}{c}$ amount of debt and gains a continuation value converging to $V_e^{+\Delta\to 0}(c)$. The instantaneous payments of tax and coupon are negligible since each time period is very short. Consequently, the firm breaks even at $\bar{f}^{Q_2}(c) \to \frac{\alpha}{c} + V_e^{+\Delta\to 0}(c) > \frac{\alpha}{c}$.

Second, given a small Δ, $\bar{f}^{Q_1}(c)$ rapidly decreases from infinity to $\frac{\alpha}{c}$ as c increases from $\bar{c}$. This is because, when c slightly exceeds $\bar{c}$, the firm must have a significant probability of default for Equation (15) to hold. In this case, with Δ being very small and the probability of any large shock approaching zero, the default boundary $\bar{f}^{Q_1}(c)$ should approximate $\frac{\alpha}{c}$ to ensure the firm defaults with a significant probability. On the other hand, when c is nearly equal to $\bar{c}$, making the present value of debt almost equal to the face value in the absence of default, the default boundary $\bar{f}^{Q_1}(c)$ must be very large to ensure minimal default risk and uphold Equation (15).

Given that $\bar{f}^{Q_2}(c)$ converges to $\frac{\alpha}{c} + V_e^{+\Delta\to 0}(c)$ and $\bar{f}^{Q_1}(c)$ rapidly decreases from infinity to $\frac{\alpha}{c}$ when $c > \bar{c}$, $\bar{f}^{Q_1}(c)$ must intersect with $\bar{f}^{Q_2}(c)$ for some $c > \bar{c}$. This implies that we can find $(\bar{f}, c)$ satisfying both Equations (15) and (16) when Δ is small.

The above intuition is illustrated in Figure 1. In this figure, the blue line represents $\bar{f}^{Q_1}(c)$, while the green line represents $\bar{f}^{Q_2}(c)$. The blue line decreases rapidly from infinity, whereas the green line decreases much more gradually. Consequently, they intersect at the point $(26.143, 0.05143)$.

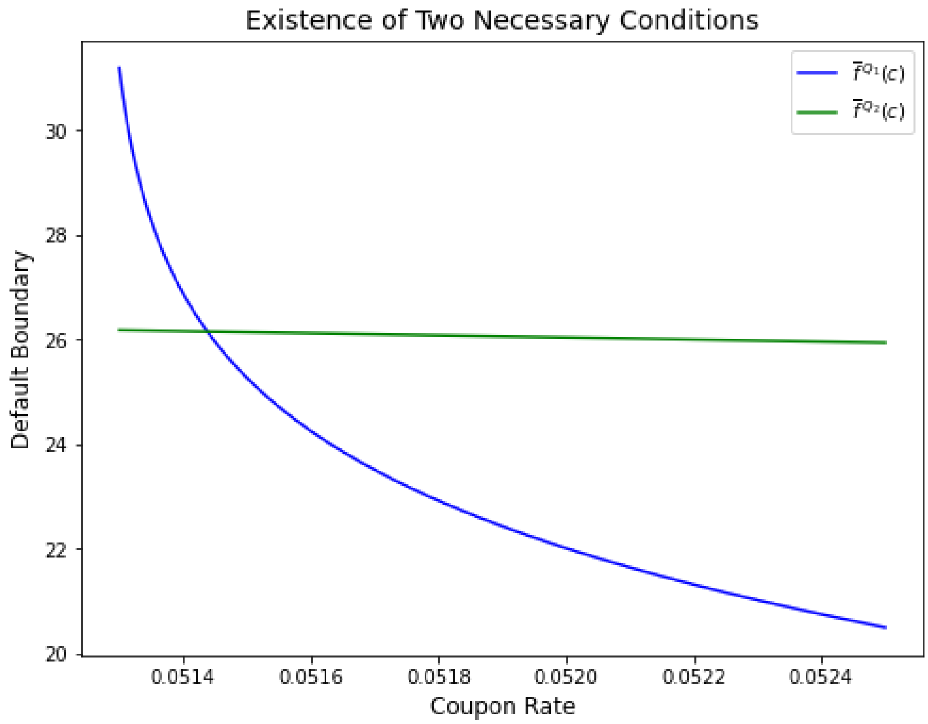

Figure 1: This figure demonstrates that there exists a pair $(\bar{f}, c)$ satisfying Equations (15) and (16). The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, $\alpha = 0.3$, and $\Delta = 1$.

Lemma 2. *We can find $\bar{\Delta}$ such that for any $\Delta < \bar{\Delta}$, there exists $(\bar{f}, c)$ such that the two necessary conditions, Equations* (15) *and* (16)*, hold.*

3.3 Existence of Equilibrium

In this section, we are going to verify that as long as $(\bar{f}, c)$ satisfy the necessary conditions, the conjectured strategies indeed constitute an equilibrium. We do this by backward induction in each stage.

First, the firm's strategy, described by Equation (1c), is optimal. This equation specifies that the firm will honor the demand clause if and only if the payment of the demand clause is smaller than its continuation value. In this case, the firm's behavior maximizes its payoff given the strategies of the debt-holders, confirming that it is optimal.

Second, let us verify that the debt-holders' strategy is optimal. When $\tilde{f}_t \leq \frac{\alpha}{c}$, the debt-holders know that the firm will not default, and the debt level at the end of the period, f_t^+, equals $\tilde{f}_t$ since no demand clause will be exercised according to the equilibrium strategies. As a result, the debt has a price $P(f_t^+) = P(\tilde{f}_t) \geq P(\frac{\alpha}{c}) = 1$ according to Equation (14) and the first necessary condition, Equation (15). This justifies the debt-holders' strategy not to exercise the demand clause, and they receive $P(f_t^+)$ by holding the debt. When $\tilde{f}_t > \frac{\alpha}{c}$, the debt-holders understand that, by the equilibrium strategy of other debt-holders, the debt level at the end of this period after exercising the demand clause will be $\frac{\alpha}{c}$. If the firm honors the demand clause, the debt price equals $P(\frac{\alpha}{c}) = 1$, which is the same as the face value. If the firm does not honor the demand clause, it goes bankrupt, and the debt-holders do not receive anything either by holding the debt or by exercising the demand clause. As a result, the debt-holders are indifferent between exercising the demand clause or not, which justifies the conjectured mixed strategy.

Third, let us verify that Equation (1b) of the firm's strategy is optimal. Before formally doing so, let us first establish a lemma that shows that issuing too low of a debt level, γ_t, is sub-optimal.

Lemma 3. *Any deviation where $\gamma_t < \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$ is dominated by $\gamma_t' = \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$ when $\Delta \to 0$. As a result, the firm cannot benefit from a deviation where $\gamma_t < \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$ as $\Delta \to 0$.*

Now, let us formally establish that the firm does not have any profitable deviation for γ_t. When $\exp(-\xi\Delta)f_t^- - \frac{\alpha}{c} < V_e^+\left(\frac{\alpha}{c}\right)$, the firm gets $V_e^+\left(\frac{\alpha}{c}\right) - \exp(-\xi\Delta)f_t^- + \frac{\alpha}{c} > 0$ by following its equilibrium strategy. If the firm deviates, we have the following cases:

1. If the firm deviates by setting $\gamma_t > \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$, the debt-holders will exercise the demand clause so that $d_t = \gamma_t + \exp(-\xi\Delta)f_t^- - \frac{\alpha}{c}$.

   (a) If $\gamma_t + \exp(-\xi\Delta)f_t^- - \frac{\alpha}{c} \geq V_e^+\left(\frac{\alpha}{c}\right)$, the firm will default on the demand clause. Anticipating this, the market prices $\tilde{P}_t' = 0$ for γ_t and therefore, the firm gets 0 from this deviation, which means that this deviation is not profitable.

   (b) If $\gamma_t + \exp(-\xi\Delta)f_t^- - \frac{\alpha}{c} < V_e^+\left(\frac{\alpha}{c}\right)$, the firm will not default on the demand clause. Anticipating this, the market prices $\tilde{P}_t' = 1$ for γ_t. As a result, the firm gets $\gamma_t - d_t + V_e^+\left(\frac{\alpha}{c}\right) = V_e^+\left(\frac{\alpha}{c}\right) - \exp(-\xi\Delta)f_t^- + \frac{\alpha}{c}$ from this deviation, which equals its payoff from following the equilibrium strategy.

2. If the firm deviates by setting $\gamma_t < \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$, the debt-holders will not exercise the demand clause. In Lemma 3, we will show that when $\Delta \to 0$, such a deviation essentially reduces the tax shield benefit without saving enough on distress costs. As a result, this deviation is not profitable.

When $\exp(-\xi\Delta)f_t^- - \frac{\alpha}{c} \geq V_e^+\left(\frac{\alpha}{c}\right)$, the firm will subsequently default on the demand clause and receive 0 by following its equilibrium strategy. If the firm deviates, we have the following cases:

1. If the deviation γ_t satisfies $\gamma_t + \exp(-\xi\Delta) f_t^- - \frac{\alpha}{c} \geq V_e^+\left(\frac{\alpha}{c}\right)$, the debt-holders will exercise the demand clause so that $d_t = \gamma_t + \exp(-\xi\Delta) f_t^- - \frac{\alpha}{c}$. In this case, the firm will not honor the demand clause, and anticipating this, the market prices $\tilde{P}_t' = 0$ for γ_t. As a result, the firm receives 0 from this deviation, which is the same as following the equilibrium strategy.

2. If the deviation γ_t satisfies $\gamma_t + \exp(-\xi\Delta) f_t^- - \frac{\alpha}{c} < V_e^+\left(\frac{\alpha}{c}\right)$, there are two cases:

   (a) If $\gamma_t + \exp(-\xi\Delta) f_t^- > \frac{\alpha}{c}$, the debt-holders will exercise the demand clause so that $d_t = \gamma_t + \exp(-\xi\Delta) f_t^- - \frac{\alpha}{c}$. In this case, the firm will honor the demand clause according to the equilibrium strategy and the market prices $\tilde{P}_t' = 1$ for γ_t. As a result, the firm receives $\gamma_t - d_t + V_e^+\left(\frac{\alpha}{c}\right) = V_e^+\left(\frac{\alpha}{c}\right) - \exp(-\xi\Delta) f_t^- + \frac{\alpha}{c} \leq 0$, which is also not profitable.

   (b) If $\gamma_t + \exp(-\xi\Delta) f_t^- \leq \frac{\alpha}{c}$, the debt-holders will not exercise the demand clause, and trivially, the firm will not default. We will show in Lemma 3 that such deviation at most gives the firm $V_e^+\left(\frac{\alpha}{c}\right) - \exp(-\xi\Delta) f_t^- + \frac{\alpha}{c} \leq 0$, which is not profitable when $\Delta \to 0$.

Lastly, (1a) is optimal by our second necessary condition, Equation (16). To be more specific, if the debt level at the beginning of the period is low, $f_t^- < \bar{f}$, the firm gets

$$\left(-\tau + \tau \min\{\alpha, c f_t^-\} - c f_t^-\right) \Delta + \frac{\alpha}{c} - f_t^- + V_e^+\left(\frac{\alpha}{c}\right) > Q_2(\bar{f}, c) = 0, \tag{20}$$

by honoring its various liabilities. As a result, it is optimal to do so. If the debt level at the beginning of the period is high, $f_t^- \geq \bar{f}$, the firm gets

$$\begin{aligned}&\left(-\tau + \tau \min\{\alpha, c f_t^-\} - c f_t^-\right) \Delta - (1 - \exp(-r\Delta)) f_t^- \\ &+ \max\left\{\frac{\alpha}{c} - \exp(-r\Delta) f_t^- + V_e^+\left(\frac{\alpha}{c}\right), 0\right\} \leq Q_2(\bar{f}, c) = 0,\end{aligned} \tag{21}$$

from honoring its various liabilities. As a result, it is optimal not to do so.

Since all the proposed strategies are shown to be optimal, we establish that they collectively form an equilibrium.

Proposition 1. *The strategies outlined in Section 3.1 constitute an equilibrium when $\Delta \to 0$.*

4 Discussion

The demand clause serves as an important tool to discipline the firm from accumulating excessive debt. The intuition is that if the firm holds an excessive amount of debt exceeding the targeted debt level, the debt-holders will immediately exercise the demand clause, forcing the firm to repay and subsequently retire the excess debt. This mechanism helps break the leverage ratchet effect, where the debt ratio continues to increase. There are two important issues to discuss. First, whether the demand clause requires strong conditions to implement, challenging the notion that the firm has no dynamic commitment power. Second, whether the demand clause enhances the firm's welfare, making the firm willing to adopt this clause from the outset.

4.1 Implementation Conditions

To effectively implement the demand clause, two key conditions must be met. First, debt-holders need to observe the firm's debt ratio. Second, debt-holders must have the ability to request repayment, and if the firm refuses, it must face bankruptcy. Both of these implementation conditions are widely assumed in the literature on debt pricing.

The first condition, that debt-holders can observe the debt ratio, is necessary for the market to competitively price the debt based on this ratio. If debt-holders cannot observe the debt ratio, they can no longer base their bids on it, and there would be no market forces to ensure the debt price reflects its true present value.

The second condition, allowing debt-holders to request repayment and impose bankruptcy if denied, parallels the situation where debt-holders can demand coupon and principal payments when the debt matures, leading to bankruptcy if the request is rejected. This condition is essential for the debt to have any value, as without it, the firm could avoid repayment without consequence, making it difficult to incentivize repayment of any debt.

Thus, the demand clause can be implemented as long as we have a well-functioning debt market where the debt is fairly priced, and the firm is disciplined by the threat of bankruptcy, both of which are commonly assumed in the literature.

From a practical perspective, demand clauses are widely used in reality. For example, depositors can withdraw their deposits at any time. They are likely to do so when they observe negative news about the bank, and the bank does not manage the situation properly, a scenario predicted by our equilibrium (as an off-equilibrium path action).

4.2 Welfare Analysis

As argued in the Leverage Ratchet effect, the firm loses a significant amount of equity value when it cannot restrain itself from issuing more debt. The intuition is that the firm incurs excessive bankruptcy costs as it continues to increase its leverage. For example, DeMarzo and He (2021) argues that equity holders do not benefit from issuing additional debt. The equity value without any commitment power is essentially the same as if the firm could not issue any debt.

In contrast, in the equilibrium we have constructed with the demand clause, we show that the equity value asymptotically achieves the equity value as if the firm had full commitment power when $\Delta \to 0$.

4.2.1 Asymptotic Equity Value with Full Commitment Power

Given any $\Delta < \bar{\Delta}$, let us denote $(\bar{f}, c)$ satisfy two necessary conditions as $(\bar{f}(\Delta), c(\Delta))$. By our construction of $\bar{f}^{Q_2}(c)$, we have $\bar{f}(\Delta) \to \frac{\alpha}{c(\Delta)} + V_e^{+\Delta\to 0}(c(\Delta))$, when $\Delta \to 0$. This implies that $\bar{f}(\Delta) > \frac{\alpha}{c(\Delta)}$. By Equation (15), we have $c(\Delta) \to r$ when $\Delta \to 0$. Figure 2 illustrates the convergence of $c(\Delta)$ to r as Δ approaches zero.

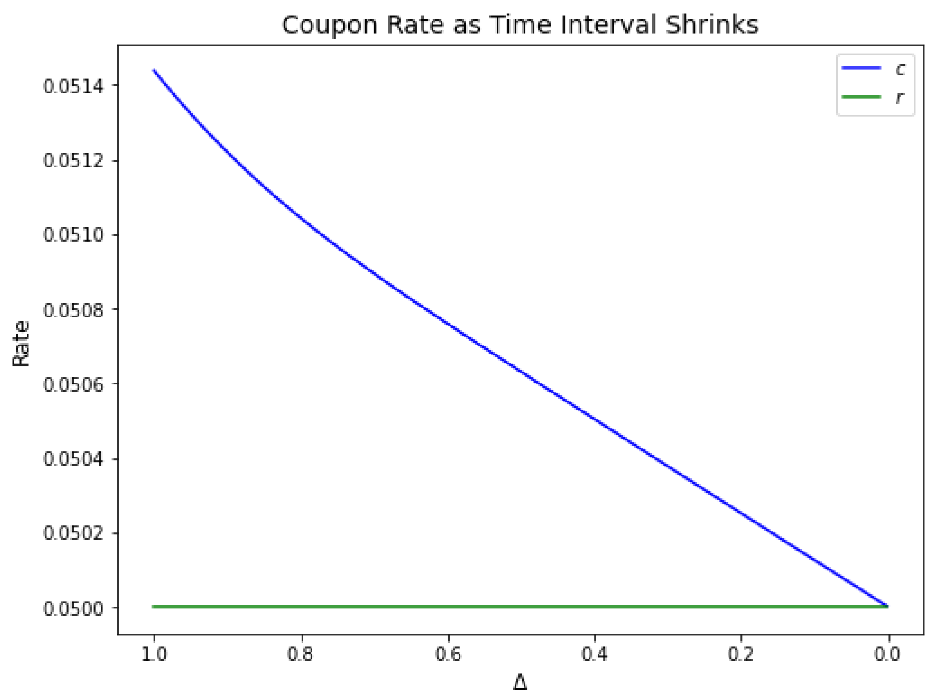

Figure 2: This figure demonstrates how $c(\Delta)$ converges to r when Δ converges to 0. The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, and $\alpha = 0.3$.

Lemma 4. *We have* $\lim_{\Delta\to 0} c(\Delta) = r$ *and* $\lim_{\Delta\to 0} \bar{f}(\Delta) = \frac{\alpha}{r} + V_e^{+\Delta\to 0}(r)$.

As a result, the equity value given $f_t^+ = \frac{\alpha}{c(\Delta)} \to \frac{\alpha}{r}$ can be calculated as:

$$\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c(\Delta)}\right) = \frac{1-(1-\alpha)\tau}{r-\mu} - \frac{\alpha}{r}. \tag{22}$$

Proposition 2. *As $\Delta \to 0$, the firm value with demandable debt converges to $\frac{1-(1-\alpha)\tau}{r-\mu}$, which represents the firm value with full tax shield benefits and zero bankruptcy costs in the limit. More specifically, the equity holders receive $\frac{\alpha}{r}$ in cash from the demandable debt issuance, and*

$$\lim_{\Delta \to 0} V_e^+ \left(\frac{\alpha}{c(\Delta)} \right) = \frac{1-(1-\alpha)\tau}{r-\mu} - \frac{\alpha}{r}$$

as the remaining equity.

Notice that $\frac{1-(1-\alpha)\tau}{r-\mu}$ represents the present value of all future after-tax income along with the tax shield per unit of current cash flow, and $\frac{\alpha}{r}$ represents the value of debt per unit of current cash flow. Because the firm can adjust its debt level frequently and adheres to the targeted debt ratio $\frac{\alpha}{c} \to \frac{\alpha}{r}$, it can fully capture the tax shield and asymptotically avoid any possibility of default. Figure 3 depicts how the default probability converges to zero when Δ converges to 0.

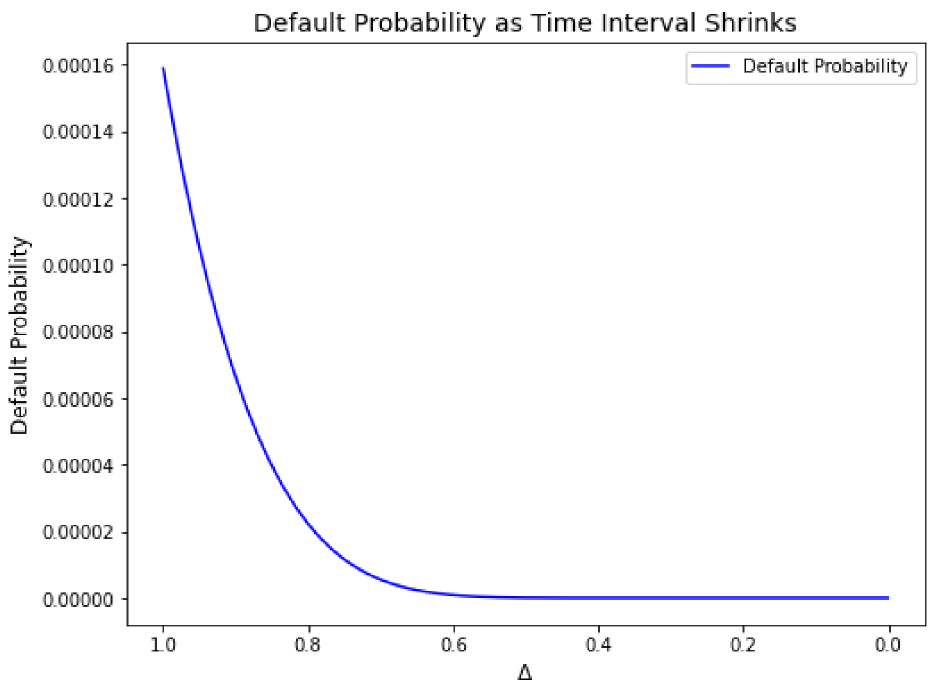

Figure 3: This figure demonstrates how the default probability converges to zero when Δ converges to 0. The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, and $\alpha = 0.3$.

Given that the initial amount of debt is issued at par value, according to our first necessary condition, Equation (15), the equity holders receive $\frac{1-(1-\alpha)\tau}{r-\mu} X_0$ from this firm, which equals the equity value as if the firm could commit to maintaining the debt level, fully achieving the tax shield, and never defaulting. Figure 4 illustrates how the firm value with demandable debt converges to the firm value as if the firm possesses full commitment power when Δ approaches zero. The blue line represents the firm's value under our mechanism with demandable debt. In contrast, the green line illustrates the firm's value assuming full commitment power, enabling the firm to fully capture the tax shield benefits while incurring zero bankruptcy costs, as derived in Proposition 2.

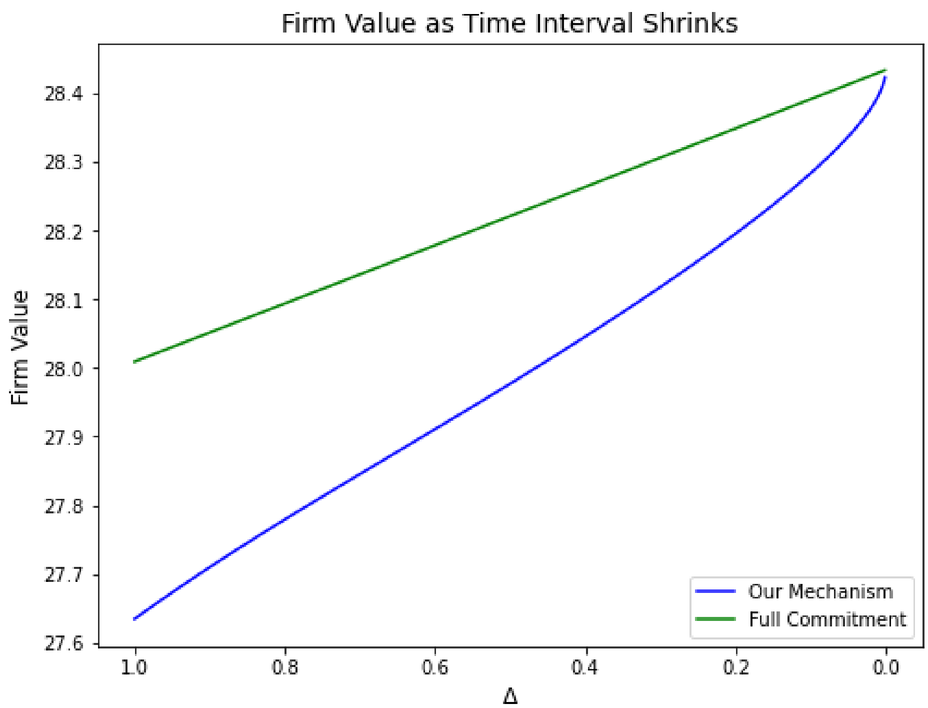

Figure 4: This figure demonstrates how the firm value with demandable debt converges to the firm value as if the firm has the full commitment power when Δ converges to 0. The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, and $\alpha = 0.3$, and $X_0 = 1$.

Since the equity value is higher with the demand clause, the firm will optimally adopt the demand clause from the outset.

4.2.2 Frequent Issuance of Debt

In the literature, the ability to frequently issue debt typically leads to a significant commitment problem and destroys firm value. As a result, if the firm can trade more frequently, it tends to lose more value, similar to the implications of the Coase Conjecture. However, this is not the case in our model.

In our case, the firm can credibly adjust its leverage ratio to the targeted debt ratio $\frac{\alpha}{c}$ as long as it avoids bankruptcy. Thus, frequent trading actually works in the firm's favor. Specifically, if the firm can trade very frequently, it can immediately respond to any cash flow shocks before those shocks accumulate. That is, when the firm experiences a positive cash flow shock, it can increase its leverage to maximize the tax shield benefit. Conversely, if the firm experiences a negative cash flow shock, it can immediately decrease its leverage before the negative shock becomes too large for the firm to handle comfortably. If the firm can trade continuously, it can fully achieve the tax shield benefits without incurring any default costs, thus reaching full efficiency.

This idea that the frequency of trading enhances efficiency is also discussed in the literature on market completeness, such as Kreps (1982) and Duffie and Huang (1985).

5 Conclusion

This paper has shown that demandable debt is an effective mechanism for addressing the leverage ratchet effect. By incorporating a demand clause, debt-holders gain the ability to discipline the firm, ensuring that the debt ratio remains within targeted levels and default risks are minimized. This approach enables firms to secure tax shield benefits while reducing the likelihood of excessive bankruptcy costs, thereby approximating the optimal outcomes observed in full commitment scenarios.

Moreover, our findings indicate that demandable debt can be implemented under conditions that are already standard in the existing literature. This insight suggests a potential evolution in how firms and regulators approach the structure of debt. Given that our results demonstrate the potential for firms to enhance their welfare through the adoption of demandable debt, such a shift could be driven by market forces.

Future research should focus on empirical evaluations of demandable debt's effectiveness. Additionally, examining variations of the model across different economic contexts would help determine its broader applicability and potential impact.

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6 Proofs

6.1 Math Preparation

In this paper, we focus on normally distributed shocks, which are widely adopted in the literature. These shocks have very nice analytical properties that serve as an important tool to smooth our analysis. Additionally, they converge to a geometric Brownian motion as the length of each period, Δ, converges to zero. This implies that it becomes increasingly unlikely to experience a large shock within one period as Δ shrinks, providing the firm an opportunity to adjust its debt level and absorb the shock. As a result, we can asymptotically achieve the equity value as if the firm had full commitment power.

By the property of normal distribution, for any $x > 0$, we have $1-\Phi(x) \in \left((1-\frac{1}{x^2})\frac{\phi(x)}{x}, \frac{\phi(x)}{x}\right)$. As a result, for any $x \to \infty$, we could write

$$1 - \Phi(x) = \frac{\phi(x)}{x}\left(1 + O(\frac{1}{x^2})\right), \tag{23}$$

where the notation $O(\frac{1}{x^2})$ refers to an order of magnitude in terms of the variable $\frac{1}{x^2}$. Similarly, for any $x \to -\infty$, we could write

$$\Phi(x) = \frac{\phi(x)}{-x}\left(1 + O(\frac{1}{x^2})\right). \tag{24}$$

Let us consider the case where $\bar{f} > \frac{\alpha}{c}$. When $\Delta \to 0$, we have

$$Z\left(\frac{\alpha}{c}\right) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{\frac{\alpha}{c}}{\bar{f}} - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{\frac{\alpha}{c}}{\bar{f}}\right) - \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{\Delta} \to -\infty \tag{25}$$

In addition, we have

$$\lim_{\Delta\to 0}\frac{\phi(Z(\frac{\alpha}{c}))}{Z(\frac{\alpha}{c})\Delta} = \lim_{\Delta\to 0}\frac{1}{\sqrt{2\pi}}\exp(-\frac{Z(\frac{\alpha}{c})^2}{2})\frac{1}{Z(\frac{\alpha}{c})\Delta} = 0. \tag{26}$$

As a result, we have $\lim_{\Delta\to 0}\frac{\Phi(Z(\frac{\alpha}{c}))}{\Delta} = 0$. By the same logic, we have for any constant A, $\lim_{\Delta\to 0}\frac{\Phi(Z(\frac{\alpha}{c})+A\sqrt{\Delta})}{\Delta} = 0$.

In addition, if there exists any $C > 0$ such that

$$\frac{\alpha}{c} < \frac{\bar{f}}{1+C},$$

we have $\lim_{\Delta\to 0}\frac{\Phi(Z(\frac{\alpha}{c}))}{\Delta}$ converges to 0 uniformly. This is true for $\lim_{\Delta\to 0}\frac{\Phi(Z(\frac{\alpha}{c})+A\sqrt{\Delta})}{\Delta}$ as well.

6.2 Proof of Lemma 1

Let us first derive $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right)$ given any c and $\bar{f}$ such that $\bar{f} > \frac{\alpha}{c}$. Let us decompose $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right)$ into several different terms. The first term is

$$\lim_{\Delta\to 0} \frac{\exp(-r\Delta)}{1-\exp\left(-(r-\mu)\Delta\right)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]}\Delta = \frac{1}{r-\mu}, \tag{27}$$

which resembles the cap rate in a perpetuity formula.

The second term is

$$\begin{aligned}
&\lim_{\Delta\to 0}\exp(\mu\Delta) - \lim_{\Delta\to 0}\tau\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]\\
&+\lim_{\Delta\to 0}\tau\alpha\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]\\
&+\lim_{\Delta\to 0}\tau\alpha\left[1-\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)\right)\right]-\lim_{\Delta\to 0}\alpha\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right]\\
&= 1-\tau+\tau\alpha-\alpha,
\end{aligned} \tag{28}$$

which represents the pre-tax income, tax, tax-shield and coupon payment.

The last term is

$$\begin{aligned}
&\lim_{\Delta\to 0}\frac{1}{\Delta}\left[-\frac{\alpha}{c}\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right]+\frac{\alpha}{c}\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]\right]\\
&=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[-\frac{\alpha}{c}+\frac{\alpha}{c}\exp(\mu\Delta)\right]=\frac{\alpha}{c}\mu,
\end{aligned} \tag{29}$$

which represents the value collected from issuing debt because the firm grows. In addition, all three terms converge uniformly when $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$.

As a result, we have $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right) = V_e^{+\Delta\to 0}(c) := \frac{1}{r-\mu}\left(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{c}\mu\right)$. The convergence is uniform for $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$.

The derivations for $\lim_{\bar{f}\to 0} V_e^+\left(\frac{\alpha}{c}\right)$, and $\lim_{\bar{f}\to\infty} V_e^+\left(\frac{\alpha}{c}\right)$ can be directly completed from the given functions. The steps are straightforward and are left as an exercise to the reader to verify.

6.3 Proof of Lemma 2

In this section, we prove that there exist $(\bar{f}, c)$ such that two necessary conditions, Equations (15) and (16), hold.

First, $Q_1(\bar{f}, c)$ is increasing in both $\bar{f}$ and c. The intuition is twofold: 1) with a larger $\bar{f}$, the firm will only default after experiencing a more significant negative shock, reducing the likelihood of default and increasing the probability that debt-holders will be paid; and 2) with a larger c, debt-holders receive more coupon payments when the firm does not default.

Furthermore, for any c satisfying Assumption 1, we have

$$\lim_{\bar{f}\to\infty} Q_1(\bar{f}, c) = \exp(-r\Delta)(c\Delta + 1) > 1, \quad \lim_{\bar{f}\to 0} Q_1(\bar{f}, c) = 0.$$

That is, if $\bar{f}$ is very large, the firm is unlikely to go bankrupt, and under Assumption 1, the present value of the debt exceeds its face value. Conversely, if $\bar{f}$ approaches zero, the firm will almost surely go bankrupt, and the present value of the debt approaches zero. By the continuity of $Q_1(\bar{f}, c)$, for any c satisfying Assumption 1, there always exists a $\bar{f}$ that satisfies Equation (15).

As a result, for any c satisfying Assumption 1, there exists a function $\bar{f}^{Q_1}(c)$ such that Equation (15) holds for the pair $(\bar{f}^{Q_1}(c), c)$. Furthermore, $\bar{f}^{Q_1}(c)$ is a decreasing and continuous function of c. Intuitively, to maintain the debt price at 1 when the debt ratio is $\frac{\alpha}{c}$, the firm must be more prone to default as the coupon payment c increases. This ensures that the expected present value of the debt remains equal to 1.

Second, by Lemma 1, we have

$$\lim_{\bar{f}\to 0} Q_2(\bar{f}, c) = -\tau\Delta + \frac{\alpha}{c} + \exp(-(r-\mu)\Delta)\Delta, \tag{30}$$

which is greater than 0 given any $\Delta \leq \bar{\Delta}'$ for some $\bar{\Delta}'$. Intuitively, when $\bar{f}$ is too small, implying a small debt burden, the firm should strictly prefer to honor its various liabilities given $\bar{f}$.

By Lemma 1, we have $\lim_{\bar{f}\to\infty} V_e^+\left(\frac{\alpha}{c}\right)$ bounded. As a result,

$$\lim_{\bar{f}\to\infty} Q_2(\bar{f}, c) \to -\infty. \tag{31}$$

Intuitively, when $\bar{f}$ is too large, implying a large debt burden, the firm should strictly prefer to default.

By continuity of $Q_2(\bar{f}, c)$, for any c, we can define a function $\bar{f}^{Q_2}(c)$ such that $(\bar{f}^{Q_2}(c), c)$ solves Equation (16). That is, there exists a medium level $\bar{f}$ such that the firm is indifferent between honoring its liabilities and defaulting. In addition, $\bar{f}^{Q_2}(c)$ can be continuous in c given the continuity of $Q_2(\bar{f}, c)$.

In addition, for any $\bar{f}$ and c such that $\bar{f} > \frac{\alpha}{c}$, we have

$$\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right) = V_e^{+\Delta\to 0}(c) := \frac{1}{r-\mu}\left[(1-\alpha)(1-\tau) + \frac{\alpha}{c}\mu\right] > 0. \tag{32}$$

This convergence is uniform for $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$.

As a result,

$$\lim_{\Delta\to 0} Q_2(\bar{f}, c) = \frac{\alpha}{c} - \bar{f} + V_e^{+\Delta\to 0}(c), \tag{33}$$

and this convergence is uniform for $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$.

For any $c \geq r$ and $\epsilon \in (0, \frac{1}{2})$, let us consider $\bar{f}' = \frac{\alpha}{c} + (1-\epsilon)V_e^{+\Delta\to 0}(c)$ and $\bar{f}'' = \frac{\alpha}{c} + (1+\epsilon)V_e^{+\Delta\to 0}(c)$. As a result, $\bar{f}'' > \bar{f}' > (1 + (1-\epsilon)\frac{1}{r-\mu}\mu)\frac{\alpha}{c}$, and we have $Q_2(\bar{f}, c)$ converges to $\frac{\alpha}{c} - \bar{f} + V_e^{+\Delta\to 0}(c)$ uniformly in $c \geq r$ and $\bar{f} \in [\bar{f}', \bar{f}'']$. This implies that $\lim_{\Delta\to 0} Q_2(\bar{f}', c) = \epsilon V_e^{+\Delta\to 0}(c) > 0$ and $\lim_{\Delta} Q_2(\bar{f}'', c) = -\epsilon V_e^{+\Delta\to 0}(c) < 0$.

As a result, there exists some $\bar{\Delta}'' > 0$ such that, for any $\Delta < \bar{\Delta}''$, we can construct

$$\bar{f}^{Q_2}(c) \in (\bar{f}', \bar{f}'') = \left(\frac{\alpha}{c} + (1-\epsilon)V_e^{+\Delta\to 0}(c), \frac{\alpha}{c} + (1+\epsilon)V_e^{+\Delta\to 0}(c)\right). \tag{34}$$

In order to establish that there exist $(\bar{f}, c)$ such that two necessary conditions hold, we just need to show that there exists c such that $\bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$. We are going to show that there exists some $\bar{\Delta}$ such that we can find the qualifying c for any $\Delta < \bar{\Delta}$. To show that, we first establish that $\bar{f}^{Q_1}(c) > \bar{f}^{Q_2}(c)$ for c close to $\bar{c}$, and $\bar{f}^{Q_1}(c) < \bar{f}^{Q_2}(c)$ for much larger c. Then we conclude that these two functions must intersect.

To be more specific, given any $c' > r$ and $\bar{f} = \frac{\alpha}{c'} + \frac{1}{2}V_e^{+\Delta\to 0}(c')$, we have

$$\begin{aligned}
&\lim_{\Delta\to 0} \frac{Q_1(\bar{f}, c') - 1}{\Delta} \\
&= \lim_{\Delta\to 0} \frac{\exp(-r\Delta)(c'\Delta + 1) - 1}{\Delta}\left(1 - \Phi\left(Z\left(\frac{\alpha}{c'}\right)\right)\right) - \lim_{\Delta\to 0} \frac{\Phi\left(Z\left(\frac{\alpha}{c'}\right)\right)}{\Delta} \\
&= c' - r > 0.
\end{aligned} \tag{35}$$

Therefore, for any $c' > r$, we can have $\bar{\Delta}(c')$ such that for any $\Delta < \bar{\Delta}(c')$, Assumption 1 is satisfied and

$$\bar{f}^{Q_1}(c') < \frac{\alpha}{c'} + \frac{1}{2}V_e^{+\Delta\to 0}(c') < \frac{\alpha}{c'} + (1-\epsilon)V_e^{+\Delta\to 0}(c') < \bar{f}^{Q_2}(c'). \tag{36}$$

Intuitively, when $\Delta \to 0$, $\bar{f}^{Q_1}(c')$ should be very close to $\frac{\alpha}{c'}$. Otherwise, given that it is very unlikely to have a large negative shock and $c' > r$, the present value of the debt is larger than 1. On the other hand, $\bar{f}^{Q_2}(c')$ converges to $\frac{\alpha}{c'} + V_e^{+\Delta\to 0}(c')$. Therefore, $\bar{f}^{Q_2}(c') > \bar{f}^{Q_1}(c')$.

In addition, $Q_1(\bar{f}, \bar{c}) < 1$ for any $\bar{f}$ and Δ. Therefore, for any $\Delta < \bar{\Delta}(c')$ and $\bar{f} = \frac{\alpha}{r} + \frac{3}{2}V_e^{+\Delta\to 0}(r)$, we can find $c'' \to \bar{c}$ such that $Q_1(\bar{f}, c'') < 1$. As a result, we have

$$\bar{f}^{Q_1}(c'') > \frac{\alpha}{r} + \frac{3}{2}V_e^{+\Delta\to 0}(r) > \frac{\alpha}{c''} + (1+\epsilon)V_e^{+\Delta\to 0}(c'') > \bar{f}^{Q_2}(c''). \tag{37}$$

Intuitively, when $c'' \to \bar{c}$, the firm should be very unlikely to default to keep the present value of debt equal to 1. In this case $\bar{f}^{Q_1}(c'')$ should be way larger than $\bar{f}^{Q_2}(c'')$ which is close to $\frac{\alpha}{c''} + V_e^{+\Delta \to 0}(c'')$.

By the continuity of $\bar{f}^{Q_1}(c)$ and $\bar{f}^{Q_2}(c)$, we can have $c \in (c'', c')$ such that $\bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$. As a result, we can find $\bar{\Delta}$ such that for any $\Delta < \bar{\Delta}$, there exists $(\bar{f}, c)$ such that Equations (15) and (16) hold.

6.4 Proof of Lemma 3

Proof. For notation purpose, let us denote $f^+_{\gamma_t} := \exp(-\xi\Delta) f^-_t + \gamma_t$. By deviating to $\gamma_t < \frac{\alpha}{c} - \exp(-\xi\Delta) f^-_t$, the firm gets

$$\begin{aligned}
&\gamma_t P(f^+_{\gamma_t}) + V_e^+\left(f^+_{\gamma_t}\right) = \gamma_t \exp(-r\Delta)\left[1 - \Phi\left(Z(f^+_{\gamma_t})\right)\right](c\Delta + 1) \\
&+ \exp(-r\Delta)\Bigg[\Delta\exp(\mu\Delta) - \tau\Delta\exp(\mu\Delta)\left[1 - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right] \\
&+ \tau\alpha\Delta\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right) - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right] \\
&+ \tau c f^+_{\gamma_t}\Delta\left[1 - \Phi\left(\tilde{Z}(f^+_{\gamma_t})\right)\right] - \left(c f^+_{\gamma_t}\Delta + f^+_{\gamma_t}\right)\left[1 - \Phi\left(Z(f^+_{\gamma_t})\right)\right] \\
&+ \left[\frac{\alpha}{c} + V_e^+\left(\frac{\alpha}{c}\right)\right]\exp(\mu\Delta)\left[1 - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right]\Bigg] \\
&= \exp(-r\Delta)\Bigg[\Delta\exp(\mu\Delta) - \tau\Delta\exp(\mu\Delta)\left[1 - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right] \\
&+ \tau\alpha\Delta\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right) - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right] \\
&+ \tau c f^+_{\gamma_t}\Delta\left[1 - \Phi\left(\tilde{Z}(f^+_{\gamma_t})\right)\right] - (c\Delta + 1)\exp(-\xi\Delta) f^-_t\left[1 - \Phi\left(Z(f^+_{\gamma_t})\right)\right] \\
&+ \left[\frac{\alpha}{c} + V_e^+\left(\frac{\alpha}{c}\right)\right]\exp(\mu\Delta)\left[1 - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right]\Bigg]
\end{aligned} \tag{38}$$

Let us denote $f^+_{\gamma'_t} := \exp(-\xi\Delta) f^-_t + \gamma'_t$. By adopting γ'_t, the firm gets $\gamma'_t P(f^+_{\gamma'_t}) + V_e^+\left(f^+_{\gamma'_t}\right)$.

Let us calculate the difference of the firm's value between adopting γ'_t and γ_t when $\Delta \to 0$. To be more specific, let us calculate

$$\lim_{\Delta \to 0} \frac{\gamma'_t P(f^+_{\gamma'_t}) + V_e^+\left(f^+_{\gamma'_t}\right) - \left(\gamma_t P(f^+_{\gamma_t}) + V_e^+\left(f^+_{\gamma_t}\right)\right)}{\Delta} = \tau c\left(\frac{\alpha}{c} - f^+_{\gamma_t}\right) > 0 \tag{39}$$

As a result, we show that γ_t is dominated by γ'_t when $\Delta \to 0$. ∎

6.5 Proof of Lemma 4

By the proof of Lemma 2, we have $\lim_{\Delta\to 0} f(\Delta) = \frac{\alpha}{c(\Delta)} + V_e^{+\Delta\to 0}(c(\Delta))$ because of construction of $\bar{f}^{Q_2}(c)$.

In addition, for any $\bar{c}' > r$, there exists a $\bar{\Delta}$ such that for all $\Delta < \bar{\Delta}$ and $c \geq \bar{c}'$, $Q_1(\frac{\alpha}{c} + V_e^{+\Delta\to 0}(c), c) > 1$. This implies that for all $\Delta < \bar{\Delta}$, $c(\Delta) \in (r, \bar{c}')$. As a result, when $\Delta \to 0$, $c(\Delta) \to r$, and $\bar{f}(\Delta) \to \frac{\alpha}{r} + V_e^{+\Delta\to 0}(r)$.

6.6 Proof of Proposition 2

Given any $c(\Delta) \to r$ and $\bar{f}(\Delta) \to \frac{\alpha}{r} + V_e^{+\Delta\to 0}(r)$, we have

$$\frac{\frac{\alpha}{c(\Delta)}}{\bar{f}(\Delta)} \to \frac{\frac{\alpha}{r}}{\frac{\alpha}{r} + \frac{1}{r-\mu}(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{r}\mu)} < \frac{1}{1+\frac{\mu}{r-\mu}}. \tag{40}$$

This implies that

$$Z\left(\frac{\alpha}{c(\Delta)}\right) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{\frac{\alpha}{c(\Delta)}}{\bar{f}(\Delta)} - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right) \to -\infty. \tag{41}$$

Importantly, this converges at the same speed as $\frac{1}{\sqrt{\Delta}}$.

As a result, we have the probability of bankruptcy converges to 0 very fast. That is,

$$\lim_{\Delta\to 0}\frac{\Phi\left(Z\left(\frac{\alpha}{c(\Delta)}\right)\right)}{\Delta} = 0.$$

Similarly, we have

$$\lim_{\Delta\to 0}\frac{\Phi\left(Z\left(\frac{\alpha}{c(\Delta)}\right) + A\sqrt{\Delta}\right)}{\Delta} = 0$$

for any A. As before, we can decompose $V_e^+\left(\frac{\alpha}{c}\right)$ into three parts and derive

$$\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c(\Delta)}\right) = \frac{1}{r-\mu}(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{r}\mu) = \frac{1-(1-\alpha)\tau}{r-\mu} - \frac{\alpha}{r}. \tag{42}$$

In addition, the firm issues $\frac{\alpha}{c(\Delta)} \to \frac{\alpha}{r}$ amount of debt at the price of 1 for any unit of the cash flow. As a result, the firm value per unit of cash flow is $\frac{1-(1-\alpha)\tau}{r-\mu}$.

In addition, let us denote the firm's value, given the time period Δ, when the firm can commit to maintaining a debt level and no default as $V^c(\Delta)$. Since the firm is able to get at most $(1-(1-\alpha)\tau)\, X_t\Delta$ at time t, therefore

$$
\begin{aligned}
V^c(\Delta) &= \mathbb{E}\left\{\sum_{t=1} \frac{1}{(1+r\Delta)^t}\,(1-(1-\alpha)\tau)\, X_t\Delta | X_0\right\} \\
&= \sum_{t=1} \exp(-r\Delta t)\,(1-(1-\alpha)\tau)\, X_0 \exp(\mu\Delta t)\Delta \\
&= (1-(1-\alpha)\tau)X_0\Delta \frac{\exp\left((\mu-r)\Delta\right)}{1-\exp\left((\mu-r)\Delta\right)}
\end{aligned}
\tag{43}
$$

Therefore, we have

$$
\lim_{\Delta\to 0} V^c(\Delta) = \frac{(1-(1-\alpha)\tau)X_0}{r-\mu}. \tag{44}
$$

Therefore, we conclude that the firm asymptotically achieves its full commitment value as the length of the time period approaches zero.