Demandable Debt and Leverage Ratchet Effect*

Kostas Koufopoulos† Xuyuanda Qi‡ Giulio Trigilia§

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Abstract

In this paper, we demonstrate that demandable debt provides an effective solution to the leverage ratchet effect without requiring any additional information beyond that assumed in the existing literature. Demandable debt-holders have an option to request full repayment of debt at any time. If the firm's leverage exceeds its target debt ratio, debt-holders will exercise their option and sell this excess debt back to the firm. This mechanism efficiently disciplines the firm to maintain the target debt ratio, except under extreme negative shocks leading to inevitable bankruptcy. Furthermore, we show that as the model's time intervals shorten, the firm can asymptotically achieve the full tax shield benefits without incurring any bankruptcy risk.

Keywords: Capital Structure, Demandable Debt, Leverage Ratchet Effect

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†University of Sussex

‡NYU Shanghai

§University of Rochester

1 Introduction

Admati, DeMarzo, Hellwig, and Pfleiderer (2018) suggest that when a firm cannot commit to future leverage choices, it tends to resist reductions in leverage, even when reducing leverage would increase firm value. In contrast, the firm ex-post prefers to increase leverage further, which can decrease firm value. Admati and coauthors call this force the *leverage ratchet effect*,[1] arguing that it arises from an agency conflict between equity-holders and debt-holders. As increasing leverage makes the existing debt riskier, thereby reducing its market value, it may benefit equity-holders, even if such an increase ultimately destroys overall firm value. Conversely, equity holders stand to lose from leverage reductions, as the benefits accrue disproportionately to the bond-holders. Thus, the firm's equilibrium leverage rises above the optimal leverage predicted by static trade-off theory, leading to higher default costs. At the extreme, under the leverage policy described in DeMarzo and He (2021), the increment in expected default costs completely offsets the tax-shield benefits of issuing any debt.

Notably, the existing papers on the leverage ratchet effect restrict attention to straight debt, as the only instrument that is tax exempt. However, in practice many other forms of debt, which for instance include covenants or optionality, are routinely issued by firms and generate tax shields. In this paper, we ask whether other forms of debt can be used to mitigate the leverage ratchet effect without additional information requirements beyond those already assumed in the existing literature. We find that *demandable debt*, or demand deposits, are an effective solution to the leverage ratchet effect.[2]

In our analysis, we demonstrate that giving debt-holders the right to sell the debt back to the issuing firm at face value disciplines the firm and prevents excessive debt issuance. With demandable debt, the firm maintains a more conservative leverage ratio, which lowers default risk and benefits equity holders, as the tax shield advantages now outweigh the expected costs of default. Importantly, to enforce the demand clause one needs the exact same information structure required for straight debt, such as in Leland (1994). That is, enforcing the demand clause only requires: 1) that the leverage ratio is publicly observable, a standard assumption in the literature for debt pricing based on leverage; and 2) that bond-holders can demand the repayment of their principal invested at any time, triggering bankruptcy if the firm refuses.

We construct a dynamic model with discrete yet short time intervals to precisely capture the timing and information available for all firm actions. In our model, the firm operates a cash-generating asset that produces taxable cash flows influenced by normally distributed shocks and faces the classical trade-off problem of optimal capital structure. The firm can

[1]The concept of a leverage ratchet effect builds on earlier studies such as Black and Scholes (1973), Bizer and DeMarzo (1992), and Leland (1994).

[2]From now on, we are going to use demandable debt and demand deposits interchangeably.

issue demandable debt with tax-deductible coupon payments, but it retains the option to strategically default on any payment—a decision that incurs significant costs and, for simplicity, results in the cessation of the asset's operations. The firm acts in the interest of its equity holders.

The intuition behind the optimality of demandable debt is as follows: as leverage rises, the debt's market price falls, due to the presence of default costs. Eventually, the price of debt will fall below its face value, prompting the debt-holders to exercise their demand clause and sell their debt back to the firm. Anticipating this response by the debt-holders, the firm is deterred from increasing its leverage above the target level. This mechanism results in a target debt ratio that always maintains the debt price at its face value, except when an extreme negative shocks precipitates the firm's default. This target ratio provides two main advantages to the firm. First, as the firm's asset base expands, it can issue additional debt to harness the tax shield benefits linked to asset growth. Second, by honoring the demand clause and adjusting its leverage following moderate cash-flow shocks, the firm reduces its leverage before these adverse shocks accumulate, thereby mitigating its risk of default.

The discreteness of time in our baseline model allows us to study the consequences of shortening the time intervals on firm value and default risk. We find that, as the time intervals shorten, the firm improves its capability to respond to shocks with near-instantaneous adjustments to its leverage level. This allows the firm to increase debt immediately after a positive cash flow shock, in order to capture nearly the full tax-shield benefits, and to promptly decrease leverage to minimize default risk after negative shocks. Asymptotically, as time becomes continuous, the firm achieves its full tax shield benefits without incurring default costs, reaching a valuation comparable to that of a fully committed, non-defaulting entity. Thus, we conclude that the leverage ratchet effect arises due to the restriction to straight debt, not due to the limited commitment friction per se.

Relation to the Literature. Our paper contributes to the literature on addressing the leverage ratchet effect. Starting from the seminal paper by DeMarzo and He (2021), both Qi (2018) and Malenko and Tsoy (2020) rely on non-Markov equilibria to resolve the leverage ratchet effect. Specifically, these papers use the equilibrium similar to that described in DeMarzo and He (2021) as a self-sustaining credible threat to punish any deviation from the target leverage policy. As a result, if the firm follows the target leverage policy, the threat is not used, allowing the firm to maintain a higher equity value. However, if the firm deviates from the target leverage policy, the self-sustaining credible threat is used, which severely decreases equity value. Rather than using an equilibrium that is highly history-dependent, our paper demonstrates that the leverage ratchet effect can be resolved within the framework of a classic Markov equilibrium.

Donaldson, Koont, Piacentino, and Vanasco (2024) explores how a credit line can prevent a firm from issuing additional debt. Their paper proposes that a credit line leads to what they term the "ratchet anti-ratchet effect," which effectively restricts the firm from leveraging further. Specifically, the lender anticipates that acquiring additional loans from the firm will incentivize the firm to draw on the credit line, which would significantly dilute the value of the additional loans. Consequently, the lender is unwilling to purchase these additional loans at a price acceptable to the firm, thereby preventing further leverage. However, while the credit line is effective in preventing leverage increases, it does not ensure that the firm can efficiently adjust its debt level—a critical capability when cash flow is volatile rather than constant, as assumed in their paper. In contrast, the demandable debt in our paper enables efficient debt adjustment. For instance, if the firm experiences negative cash flow, demandable debt incentivizes the firm to repurchase debt to mitigate excessive bankruptcy risk, whereas the credit line discussed in Donaldson et al. (2024) cannot provide this flexibility, and works in their environment only because the firm's cash flows are constant.

This paper also contributes to the literature on the adoption and implications of demandable debt. A significant focus in financial economics has been the role of demandable debt in providing liquidity insurance to investors when liquidity shocks are non-contractible. Diamond and Dybvig (1983) underscores demandable debt can effectively offer such insurance. Extending this discourse, Jacklin (1987) examines the conditions under which demandable debt outperforms dividend-paying securities in mitigating liquidity risks, emphasizing that the effectiveness hinges on the presence of trading restrictions. Further, Diamond and Rajan (2001) argues that demandable debt enables relationship lenders to commit to utilizing their specialized skills for collecting returns even after transferring holdings to other investors, thereby enhancing liquidity without sacrificing performance. Additionally, Calomiris and Kahn (1991) illustrates that demandable debt can attract deposits by giving creditors the option to force liquidation when there is a risk of bankers misappropriating funds. In such cases, debt-holders, upon receiving adverse signals, may exercise early withdrawal to safeguard their investments. Building on these insights, our paper introduces a novel function of demandable debt: aiding firms in sustaining an optimal capital structure and curbing excessive debt issuance. This mechanism offers a practical tool for aligning the interests of shareholders and debt-holders, contributing to financial stability by preventing over-leverage and mitigating default risks.

Our paper is also related to the literature on how trading frequency can enhance welfare. For example, Kreps (1982) and Duffie and Huang (1985) discuss how more frequent portfolio re-balancing allows investors to construct Arrow-Debreu securities with fewer assets. Our comparative static as time-intervals shorten shares a similar flavor: shorter period lengths and more frequent re-balancing imply that shocks in each period are likely smaller, allowing the firm to adjust its leverage ratio more promptly to capture the tax shield benefit and avoid default costs, resulting in a more efficient outcome. This contrasts with the standard intuition in commitment problems such as Coase (1972), where a shorter time period reduces the firm's commitment power, leading to greater welfare losses.

The paper is structured as follows. Section 2 presents the economic environment and the equilibrium concept. Section 3 describes the conjectured equilibrium strategies, pins down two necessary conditions of the equilibrium, and proves the existence of equilibrium. Section 4 discusses the implementation conditions and the welfare implication of using demandable debt. Section 5 concludes.

2 Model Setup

2.1 Economic Environment

We analyze a discrete-time model with periods indexed by $t = 0, 1, 2, \ldots$. All players are risk-neutral and discount future payoffs using the factor $\exp(-r\Delta)$ per period, where Δ represents the length of each time period.

The firm, acting in the interest of its shareholders, operates a cash-generating machine that produces a pre-tax cash flow of $X_t\Delta$ at time $t \geq 1$. The cash flow level X_t follows a log-normal distribution, given by:

$$X_t = g(Z_t)X_{t-1}, \tag{1}$$

where $g(Z_t) = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta}Z_t\right)$ such that $\mu < r$ is the percentage drift, σ is the percentage volatility, and Z_t is a standard normal random variable. The cash flow is subject to taxation at a rate τ.

After observing and collecting the pre-tax cash flow, the firm can issue new long-term debt or repurchase non-maturing long-term debt at any date $t \geq 0$ in a competitive financial market. Let us denote the amount of newly issued or repurchased debt in period t as Γ_t, where $\Gamma_t > 0$ indicates that new debt is issued, and $\Gamma_t < 0$ indicates that non-maturing long-term debt is repurchased in this period. This debt pays a coupon each period at a rate of $c\Delta$ and matures at a rate of $1 - \exp(-\xi\Delta)$ starting from the next period after issuance. The coupon payments are tax-deductible up to α fraction of the cash flow. We assume that c and ξ are chosen so that the following condition holds:

Assumption 1. $c > \bar{c} := \frac{\exp(r\Delta)-1}{\Delta}$.

Violation of Assumption (1) implies that the present value of the debt remains below its face value even when the firm has a negligible chance of default, prompting all debt-holders to exercise the demand clause—discussed later—resulting in the trivial outcome where no debt remains outstanding when the debt is competitively priced. To see this, note that the discounted sum of all payments to a debt-holder with a unit face value is bounded by

$$\begin{aligned}&\sum_{t\geq 1} \exp(-(r+\xi)\Delta t) \exp(\xi\Delta)\left(c\Delta + 1 - \exp(-\xi\Delta)\right) \\ &= \exp(-r\Delta)\left(c\Delta + 1 - \exp(-\xi\Delta)\right) \frac{1}{1 - \exp\left(-(r+\xi)\Delta\right)},\end{aligned}$$

which represents the present value of the debt assuming no default, and is less than 1 if Assumption (1) fails.

The key distinction of this paper from the classical literature is the inclusion of a demand clause in the debt, which grants the debt-holder the option to demand full repayment of the principal at any time after the debt has been issued. Specifically, at any date t, after observing the realization of the cash flow level and the newly issued or repurchased amount of debt, a debt-holder with non-maturing debt can request the firm to repay the face value of the debt early, in addition to the coupon payment for that period. Once the firm honors the early repayment, the debt is put back to the firm and immediately retired. Following the literature, we assume that each debt-holder is atomless, making each one a price taker who cannot change the debt level by exercising the demand clause individually. However, the aggregation of the debt-holders' actions will change the debt level. We denote the amount of debt for which the debt-holders have exercised the demand clause at date t as D_t.

Let us denote the outstanding level of debt at the end of period t as F_t. Since the outstanding level of debt at the beginning of period t equals the outstanding level at the end of the previous period, we know that the outstanding debt at the beginning of period t is F_{t-1}. In addition, we assume that the firm does not have any debt outstanding at the beginning of $t = 0$, prior to the initial issuance. With a slight abuse of notation, we denote the amount of debt outstanding at the beginning of $t = 0$ as $F_{0-} := 0$. Given that debt matures at a rate $1 - \exp(-\xi\Delta)$, the firm issues or repurchases Γ_t, and D_t is put back by the debt-holders through the exercise of the demand clause, the dynamics of the outstanding debt can be expressed as:

$$F_t = \exp(-\xi\Delta) F_{t-1} + \Gamma_t - D_t. \tag{2}$$

The firm has the strategic option to default, meaning it can choose to forgo its tax obligations to the government and debt payments to debt-holders. These debt payments include coupon payments, maturing debt, and any debt that is put back due to debt-holders exercising the demand clause. In this case, the firm declares bankruptcy and incurs significant bankruptcy costs. For simplicity, we assume that bankruptcy costs equal the continuation value of the firm going forward. That is, the firm forfeits all future cash flows after it declares bankruptcy.

To summarize, we can decompose any period t into the following four stages:

1. The current cash flow level X_t is realized and publicly observed. The firm collects $X_t\Delta$ as pre-tax profit.

2. The firm determines the amount of debt to issue or repurchase, denoted as Γ_t.

3. Non-maturing debt-holders (including those who just purchased the debt in this period) decide whether to exercise the demand clause, which we denote as $\mathbb{1}_t^d$, where $\mathbb{1}_t^d = 1$ indicates that the demand clause is exercised, and $\mathbb{1}_t^d = 0$ indicates that it is not. The aggregate amount of debt put back in this period is denoted as D_t.

4. The firm decides whether to default, which we denote as $\mathbb{1}_t^b$, where $\mathbb{1}_t^b = 1$ indicates that the firm honors its obligations, and $\mathbb{1}_t^b = 0$ indicates that it defaults.

   (a) If the firm honors its obligations:

   - It pays the coupon for the existing debt, $cF_{t-1}\Delta$,
   - It pays taxes $\tau X_t\Delta - \tau \min(\alpha X_t, cF_{t-1})\Delta$,
   - It repays the maturing debt $(1 - \exp(-\xi\Delta))F_{t-1}$,
   - It repays the put-back debt D_t.

   After these payments, the game moves to the next period.

   (b) If the firm declares bankruptcy:

   - The game ends with zero continuation value for all players.

2.2 Equilibrium Concept

2.2.1 Strategy

In this paper, we focus on the Markov Perfect Equilibrium (MPE). Formally, at any date t, the firm determines its debt issuance or repurchase policy Γ_t based on the current cash flow level X_t and the previous period's outstanding debt F_{t-1}. That is, $\Gamma_t = \Gamma(X_t, F_{t-1})$ is a random variable over the interval $[-\exp(-\xi\Delta)F_{t-1}, \infty)$, where the lower bound ensures that the firm cannot repurchase more debt than the outstanding non-maturing amount.

After observing the realization of ${\Gamma_t}^3$, together with X_t and F_{t-1}, non-maturing debt-holders decide whether to exercise the demand clause at date t. Formally, their decision to put back the debt is denoted by $\mathbb{1}_t^d = \mathbb{1}^d(X_t, F_{t-1}, \Gamma_t)$, a random variable taking values in $\{0, 1\}$. Since each debt-holder is atomless, and by the law of large numbers, the aggregate amount of debt put back at time t, denoted by D_t, is given by:

$$D_t = D(X_t, F_{t-1}, \Gamma_t) = (\exp(-\xi\Delta)F_{t-1} + \Gamma_t)\mathbb{E}[\mathbb{1}^d \mid X_t, F_{t-1}, \Gamma_t], \tag{3}$$

where $\exp(-\xi\Delta)F_{t-1} + \Gamma_t$ represents the total amount of non-maturing debt outstanding after issuance or repurchase, and $\mathbb{E}[\mathbb{1}^d \mid X_t, F_{t-1}, \Gamma_t]$ represents the expected fraction of debt-holders who exercise the demand clause at time t.

After that, the firm then makes its default decision based on the current cash flow level X_t, the outstanding debt from the previous period F_{t-1}, the amount of newly issued or repurchased debt Γ_t, and the amount of debt put back by debt-holders D_t. Formally, the firm's default choice is represented by $\mathbb{1}_t^b = \mathbb{1}^b(X_t, F_{t-1}, \Gamma_t, D_t)$, a random variable taking values in $\{0, 1\}$.

2.2.2 Payoff

Given this Markovian structure, we can express the payoffs for equity-holders and debt-holders recursively. To be more specific, at the beginning of period t, we can denote the equity value as $V_e^-(X_t, F_{t-1})$, and market-to-par ratio of debt as $V_d^-(X_t, F_{t-1})$. Similarly, at the end of period t, conditional on the firm not declaring bankrupcty, the equity value at the end of date t as $V_e^+(X_t, F_t)$ and the market-to-par-ratio of (non-maturing) debt as $V_d^+(X_t, F_t)$.

Throughout this paper, we assume that the debt is competitively priced by the market. As a result, the debt price at the end of date t equals the fair value of debt, which depends only on X_t and F_t, conditional on the firm not declaring bankruptcy. We denote this debt price as $P(X_t, F_t) = V_d^+(X_t, F_t)$.

Given initial states (X_t, F_{t-1}), the strategies $\left(\Gamma, \mathbb{1}^d, \mathbb{1}^b\right)$, and the price policy P, let us consider the firm issues/repurchases Γ_t' amount of debt, which may possibly be a deviation. The price at which the debt is issued/repurchased is calculated based on the debt-holder's strategy of exercising the demand clause and the firm's strategy of whether to default. To be more specific, by competitive pricing, the price level of debt at which the debt is issued/repurchased at date t is:

[3]With slight abuse of notation, we use Γ_t to denote its realization as well.

$$\tilde{P}'(X_t, F_{t-1}, \Gamma'_t) = \mathbb{E}\left\{\mathbb{1}^b(X_t, F_{t-1}, \Gamma'_t, \tilde{D}'_t) P(X_t, \tilde{F}'_t) | X_t, F_{t-1}, \Gamma'_t\right\}, \tag{4}$$

where

$$\tilde{D}'_t = \left(\exp(-\xi\Delta)F_{t-1} + \Gamma'_t\right) \mathbb{E}\left[\mathbb{1}^d | X_t, F_{t-1}, \Gamma'_t\right],$$

and

$$\tilde{F}'_t = \exp(-\xi\Delta)F_{t-1} + \Gamma'_t - \tilde{D}'_t.$$

As a result, the equity value at the beginning of date t, with possible deviations $\left(\Gamma'_t, D'_t, \mathbb{1}_t^{b'}\right)$, can be written as

$$\begin{aligned} V_e^-\left(X_t, F_{t-1}, \Gamma'_t, D'_t, \mathbb{1}_t^{b'}\right) &= X_t\Delta + \tilde{P}'\Gamma'_t + \mathbb{1}_t^{b'}\Big[\big(-\tau X_t + \tau \min\{\alpha X_t, cF_{t-1}\} \\ &- cF_{t-1}\big)\Delta - \left(1 - \exp(-\xi\Delta)\right) F_{t-1} - D'_t + V_e^+(X_t, F'_t)\Big], \end{aligned} \tag{5}$$

where

$$F'_t = \exp(-\xi\Delta)F_{t-1} + \Gamma'_t - D'_t,$$

represents the debt level at the end of date t, given the potential deviation Γ'_t and D'_t.

For any debt at the beginning of date t, with probability $1 - \exp(-\xi\Delta)$, the debt matures at this period, and with probability $\exp(-\xi\Delta)$, the debt does not mature at this period. As a result, with possible deviations $\left(\Gamma'_t, D'_t, \mathbb{1}_t^{b'}\right)$, the market-to-par ratio of debt held by an atomless debt-holder, who chooses $\mathbb{1}_t^{d'}$, at the beginning of date t can be written as

$$\begin{aligned} &V_d^-\left(X_t, F_{t-1}, \Gamma'_t, D'_t, \mathbb{1}_t^{b'} \mid \mathbb{1}_t^{d'}\right) \\ &= \mathbb{1}_t^{b'}\left(c\Delta + 1 - \exp(-\xi\Delta) + \exp(-\xi\Delta)\left(\mathbb{1}_t^{d'} + (1 - \mathbb{1}_t^{d'})V_d^+(X_t, F'_t)\right)\right). \end{aligned} \tag{6}$$

By definition, the equity value and the debt value at the beginning of period t, $V_e^-(V_t, F_{t-1})$ and $V_d^-(V_t, F_{t-1})$, can be calculated from equations (5) and (6) with equilibrium strategy $\left(\Gamma, \mathbb{1}^d, \mathbb{1}^b\right)$ and D calculated from $\mathbb{1}^d$:

$$V_e^-(X_t, F_{t-1}) = \mathbb{E}\left\{V_e^-\left(X_t, F_{t-1}, \Gamma, D, \mathbb{1}^b\right) \mid X_t, F_{t-1}\right\}, \tag{7}$$

and

$$V_d^-(X_t, F_{t-1}) = \mathbb{E}\left\{V_d^-\left(X_t, F_{t-1}, \Gamma, D, \mathbb{1}^b \mid \mathbb{1}^d\right)|X_t, F_{t-1}\right\}. \tag{8}$$

The corresponding values at the end of the period can be written as the values at the beginning of the next period:

$$V_e^+(X_t, F_t) = \mathbb{E}\{\exp(-r\Delta)V_e^-(X_{t+1}, F_t)|X_t, F_t\}, \tag{9}$$

and

$$V_d^+(X_t, F_t) = \mathbb{E}\{\exp(-r\Delta)V_d^-(X_{t+1}, F_t)|X_t, F_t\}. \tag{10}$$

2.2.3 Equilibrium Conditions

By the one-shot deviation principle, the strategies $\left(\Gamma, \mathbb{1}^d, \mathbb{1}^b\right)$ construct an equilibrium if and only if for any $(X_t, F_{t-1}, F_t, \Gamma_t', D_t')$,

1. **Optimality of Equity Holder**
   - (Optimality of Debt Policy) Given any (X_t, F_{t-1}), we have

   $$V_e^-(X_t, F_{t-1}) = \max_{\Gamma_t''} \mathbb{E}\left\{V_e^-\left(X_t, F_{t-1}, \Gamma_t'', D, \mathbb{1}^b\right)|X_t, F_{t-1}, \Gamma_t''\right\} \tag{11}$$

   - (Optimality of Default Policy) Given any $(X_t, F_{t-1}, \Gamma_t', D_t')$, we have

   $$\begin{aligned}&\mathbb{E}\left\{V_e^-\left(X_t, F_{t-1}, \Gamma_t', D_t', \mathbb{1}^b\right) \mid X_t, F_{t-1}, \Gamma_t', D_t'\right\}\\ &= \max_{\mathbb{1}_t^{b''}} \mathbb{E}\left\{V_e^-\left(X_t, F_{t-1}, \Gamma_t', D_t', \mathbb{1}_t^{b''}\right) \mid X_t, F_{t-1}, \Gamma_t', D_t', \mathbb{1}_t^{b''}\right\}\end{aligned} \tag{12}$$

2. **Optimality of Debt Holder** Given any $(X_t, F_{t-1}, \Gamma_t')$, we have

   $$\begin{aligned}&\mathbb{E}\left\{V_d^-\left(X_t, F_{t-1}, \Gamma_t', D, \mathbb{1}^b \mid \mathbb{1}^d\right)|X_t, F_{t-1}, \Gamma_t'\right\}\\ &= \max_{\mathbb{1}_t^{d''}} \mathbb{E}\left\{V_d^-\left(X_t, F_{t-1}, \Gamma_t', D, \mathbb{1}^b \mid \mathbb{1}_t^{d''}\right)|X_t, F_{t-1}, \Gamma_t', \mathbb{1}_t^{d''}\right\}\end{aligned} \tag{13}$$

3. **Competitive Pricing of Debt.** Given any (X_t, F_t), we have

   $$P(X_t, F_t) = V_d^+(X_t, F_t), \tag{14}$$

   and $\tilde{P}'(X_t, F_{t-1}, \Gamma_t')$ is defined as in Equation (4).

Equation (11) describes the firm's optimal decision regarding debt issuance or repurchase, which aims to maximize equity value. Equation (12) characterizes the firm's optimal default decision. We will later show that the firm defaults optimally if and only if its various obligations exceed its continuation value. Equation (13) captures the debt-holders' optimal exercise of the demand clause. Intuitively, they will refrain from exercising the clause if the market price of debt exceeds its face value and will exercise it if the market price falls below the face value. Finally, Equation 14 states that the market price of debt at the end of the period is fairly determined by the prevailing cash flows and debt levels.

3 Equilibrium

3.1 Conjectured Equilibrium

In this paper, we show that as $\Delta \to 0$, there exists an equilibrium that depends crucially on the debt ratios $f_t^- = \frac{F_{t-1}}{X_t}$ and $f_t^+ = \frac{F_t}{X_t}$ at any time t, given a properly designed coupon rate c. Before we formally start the discussion, we normalize Γ_t and D_t in a similar way by $\gamma_t = \frac{\Gamma_t}{X_t}$ and $d_t = \frac{D_t}{X_t}$ for notational purposes.

Specifically, in this section, we show that there exists a threshold $\bar{f}$ such that the firm adjusts its debt level to maintain a target debt ratio of $\frac{\alpha}{c}$, and subsequently chooses not to default at date t if $f_t^- < \bar{f}$, and maintains the current debt level and defaults otherwise. On the other hand, the debt-holders use their demand clause as a disciplinary tool to ensure that the firm adheres to this equilibrium strategy. We demonstrate that debt-holders will exercise their demand clause if $f_{\gamma_t}^+ > \frac{\alpha}{c}$, thereby forcing the firm to reduce leverage ratio to the target level, where

$$f_{\gamma_t}^+ := \exp(-\xi\Delta) f_t^- + \gamma_t,$$

represents the post-adjustment debt ratio. Conversely, if $f_{\gamma_t}^+ \leq \frac{\alpha}{c}$, debt-holders will not exercise the demand clause.

Normalized Values

Given the conjecture that what matters for the equilibrium strategies and values are the debt ratio f_t^- and the debt ratio f_t^+, the cash flow level X_t serves only as a multiplier for equity values[4]. As a result, we can define $V_e^-(f_t^-)$, $V_d^-(f_t^-)$, $V_e^+(f_t^+)$, and $V_d^+(f_t^+)$ to represent the equity values per unit of cash flow and the debt value as functions of debt ratios as follows:

$$V_e^-(f_t^-) = \frac{V_e^-(X_t, F_{t-1})}{X_t}, \quad V_d^-(f_t^-) = V_d^-(X_t, F_{t-1}),$$

[4]Notice that we already normalize the debt value by defining it as the debt value per face value.

$$V_e^+(f_t^+) = \frac{V_e^+(X_t, F_t)}{X_t}, \quad V_d^+(f_t^+) = V_d^+(X_t, F_t).$$

Similarly, we can normalize the amount of debt issuance/repurchase and the amount of debt for which the demand clause has been exercised with respect to X_t, the firm's current cash flow. Specifically, we define:

$$\gamma(f_t^-) = \frac{\Gamma(X_t, F_{t-1})}{X_t}, \quad d(f_t^-, \gamma_t) = \frac{D(X_t, F_{t-1}, \gamma_t X_t)}{X_t}.$$

With some abuse of notation, we represent other strategies as functions of debt ratios as follows:

$$\mathbb{1}^d(f_t^-, \gamma_t) = \frac{\mathbb{1}^d(X_t, F_{t-1}, \gamma_t X_t)}{X_t}, \; \mathbb{1}^b(f_t^-, \gamma_t, d_t) = \frac{\mathbb{1}^b(X_t, F_{t-1}, \gamma_t X_t, d_t X_t)}{X_t}, \; P(f_t^+) = V_d^+(f_t^+).$$

3.1.1 Equilibrium Conjecture

With all the above notation in place, we can now formalize the equilibrium conjecture as follows:

We conjecture the threshold $\bar{f}$ to satisfy

$$\gamma(\bar{f}) - \Pi\left(\bar{f}, \gamma(\bar{f}), d\left(\bar{f}, \gamma(\bar{f})\right)\right) + V_e^+\left(\exp(-\xi\Delta)\bar{f} + \gamma(\bar{f}) - d\left(\bar{f}, \gamma(\bar{f})\right)\right) = 0, \tag{15}$$

where the function Π represents the payment burden after the debt issuance/repurchase and is calculated as

$$\Pi(f_t^-, \gamma_t, d_t) := \tau\Delta - \tau \min\left\{\alpha, c f_t^-\right\}\Delta + c f_t^-\Delta + (1 - \exp(-\xi\Delta))\, f_t^- + d_t.$$

The debt issuance policy $\gamma(f_t^-)$ and the amount of exercised demand clause $d(f_t^-, \gamma_t)$ will be formally defined in the conjectured strategies section below. We will show later that Equation (15) has important implication on the optimal decision of the firm and serves as one of two important identities which characterizes the equilibrium.

Given this conjectured threshold, we propose the following strategies as part of the conjectured equilibrium.

1. **Debt-Holders' Strategy:**

$$\mathbb{1}^d(f_t^-, \gamma_t) = \begin{cases} \begin{cases} 1 & \text{with probability } \frac{f_{\gamma_t}^+ - \frac{\alpha}{c}}{f_{\gamma_t}^+} \\ 0 & \text{with probability } \frac{\frac{\alpha}{c}}{f_{\gamma_t}^+} \end{cases}, & \text{if } f_{\gamma_t}^+ > \frac{\alpha}{c} \\ 0, & \text{if } f_{\gamma_t}^+ \leq \frac{\alpha}{c} \end{cases}$$

Debt-holders use the demand clause as a disciplinary tool, exercising it with a probability proportional to the excess of the post-adjustment debt ratio above the target $\frac{\alpha}{c}$. If the debt ratio is at or below the target, they do not exercise the demand clause.

2. **Firm's Strategies:**

   (a) **Strategy for Optimal Default:**

$$\mathbb{1}^b(f_t^-, \gamma_t, d_t) = \begin{cases} 1, & \text{if } \Pi(f_t^-, \gamma_t, d_t) < V_e^+ \left(\exp(-\xi\Delta) f_t^- + \gamma_t - d_t\right) \\ 0, & \text{otherwise} \end{cases}$$

   This strategy follows from Equation (12), which implies that the firm will choose not to default if and only if the total payment burden—comprising tax obligations and debt-related costs—is less than the continuation value of equity.

   (b) **Strategy for Debt Issuance/Repurchase:**

$$\gamma(f_t^-) = \begin{cases} \frac{\alpha}{c} - \exp(-\xi\Delta) f_t^-, & \text{if } f_t^- < \bar{f} \\ 0, & \text{if } f_t^- \geq \bar{f} \end{cases}$$

   The firm issues or repurchases debt to maintain the target debt ratio $\frac{\alpha}{c}$ whenever its current leverage is below the default threshold.

Given the above-conjectured threshold $\bar{f}$ and the associated strategies, we can readily conclude that the firm will default if and only if $f_t^- \geq \bar{f}$.

3.2 Two Supporting Conditions

To establish our conjecture as an equilibrium, we require that the coupon rate c and the conjectured threshold $\bar{f}$ satisfy two *supporting conditions*. We will subsequently demonstrate that (1) there exists a pair $(\bar{f}, c)$ such that these supporting conditions hold, and (2) given these conditions, our conjectured equilibrium indeed constitutes an equilibrium.

The first condition corresponds to Equation (13) and ensures the optimality of the debt-holders' strategy. Specifically, it guarantees that the debt is traded at par when the debt ratio equals $\frac{\alpha}{c}$, rendering the debt-holder indifferent between exercising the demand clause or continuing to hold the debt. We refer to this as the *supporting condition for optimal demand clause exercise.*

The second condition is a restatement of Equation (15) and ensures the optimality of the firm's strategy. Intuitively, the firm can always guarantee a continuation value of zero by maintaining its current debt level and subsequently defaulting. This condition ensures that, when the beginning-of-period debt ratio f_t^- is at the default threshold $\bar{f}$, the firm breaks even by repurchasing debt according to the debt issuance strategy (2b) specified in the conjectured equilibrium and choosing not to default. We refer to this as the *supporting condition for optimal default.* We will verify later that this condition guarantees the firm's indifference between defaulting and continuing when the debt ratio is exactly $\bar{f}$, taking into account the debt issuance decision and the corresponding exercise of the demand clause by debt-holders in equilibrium.

Supporting Condition for Optimal Demand Clause Exercise

In order to establish this necessary condition, it is important to conjecture the subgame equilibrium at period $t+1$, given f_t^+ at the end of date t. This characterizes the value of debt if the debt-holder chooses not to exercise the demand clause. In other words, it provides the outside option for the debt-holder and therefore plays a crucial role in determining the optimality of the decision to exercise the demand clause.

Subgame Equilibrium at Period $t+1$: Given our equilibrium conjecture, we know that the firm will not default at date $t+1$ if and only if $f_{t+1}^- < \bar{f}$. According to the debt issuance strategy (2b), the firm will issue or repurchase debt such that $\tilde{f}_{t+1} = \frac{\alpha}{c}$ at period $t+1$ if it does not default. Specifically, let us define

$$Z(f_t^+) = \frac{1}{\sigma\sqrt{\Delta}} \left(\log \frac{f_t^+}{\bar{f}} - \left(\mu - \frac{\sigma^2}{2} \right) \Delta \right).$$

Given Equation (1), the dynamics of the cash flow, we have $f_{t+1}^- < \bar{f}$ if and only if the cash flow does not experience a large negative shock. That is, $Z_{t+1} > Z(f_t^+)$. As a result, $f_{t+1}^- < \bar{f}$ occurs with probability $1 - \Phi\left(Z(f_t^+)\right)$, where $\Phi(\cdot)$ represents the normal distribution.

Deriving Supporting Condition: By our conjecture that the debt-holders will exercise the demand clause to maintain the debt ratio $f_t^+ = \frac{\alpha}{c}$ if the debt ratio after the issuance is too high $f_{\gamma t}^+ > \frac{\alpha}{c}$, and the equilibrium condition (13), the debt-holders should be indifferent to exercising the demand clause or not, given the market price $P\left(\frac{\alpha}{c}\right)$, which implies that $P\left(\frac{\alpha}{c}\right)$ equals its face value of 1.

In addition, according to the firm's conjectured strategy (2b), the firm will issue or repurchase non-maturing debt to maintain the level $\frac{\alpha}{c}$ as long as $f_{t+1}^- < \bar{f}$ at date $t+1$. As a result, any non-maturing debt at the end of date $t+1$ is also traded at par, provided the firm does not default. Therefore, the value of the debt at the beginning of date $t+1$ when $f_{t+1}^- < \bar{f}$ equals $c\Delta + 1$, where $c\Delta$ is the coupon payment and 1 is the principal value, regardless of whether the debt is repurchased, retired, or held—so long as the firm does not default.

Given any f_t^+ at the end of period t, if the firm experiences a large negative cash flow shock, $Z(f_t) \leq Z_{t+1}$, the firm will default, and the debt-holder will receive nothing; otherwise, the firm will not default. In the latter case, the debt-holder with a unit of face value will receive $c\Delta + 1$, as discussed. Given the probability of the cash flow shock, the price of debt at the end of period t is

$$P(f_t^+) = \exp(-r\Delta)\left(1 - \Phi\left(Z(f_t^+)\right)\right)(c\Delta + 1). \tag{16}$$

By Equation (16), $\bar{f}$ and c play important roles in determining the debt price. $\bar{f}$, which is the bankruptcy threshold, determines how likely the firm is to receive the next period's payments through $Z(f_t^+)$, and c, which is the coupon rate, determines the amount of payment, given there is no default. Let us denote $Q_1(\bar{f}, c) = P\left(\frac{\alpha}{c}\right)$ to explicitly express how $\bar{f}$ and c impact the debt price when the firm has $\frac{\alpha}{c}$ debt outstanding at the end of the period. Given $P\left(\frac{\alpha}{c}\right) = 1$, $\bar{f}$ and c must necessarily satisfy the following relationship:

$$Q_1(\bar{f}, c) = \exp(-r\Delta)\left(1 - \Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right)(c\Delta + 1) = 1. \tag{17}$$

Supporting Condition for Optimal Default

This supporting condition rewrites Equation (15). By plugging the conjectured debt issuance strategy (2b) and the conjectured debt-holder's strategy (1) into the left-hand side of Equation (15), we obtain the following expression, which crucially depends on $\bar{f}$ and c:

$$\begin{aligned} Q_2(\bar{f}, c) &:= \left(\frac{\alpha}{c} - \exp(-\xi\Delta)\bar{f}\right) - \left(\tau - \tau\min\left(\alpha, c\bar{f}\right) + c\bar{f}\right)\Delta - (1 - \exp(-\xi\Delta))\bar{f} + V_e^+\left(\frac{\alpha}{c}\right) \\ &= -\left(\tau - \tau\min\left(\alpha, c\bar{f}\right) + c\bar{f}\right)\Delta + \frac{\alpha}{c} - \bar{f} + V_e^+\left(\frac{\alpha}{c}\right) \end{aligned} \tag{18}$$

The quantity $Q_2(\bar{f}, c)$ carries strong economic meaning. It represents the value of the firm when it chooses not to default and adjusts its debt to the level $\frac{\alpha}{c}$, starting from an initial debt level of $\bar{f}$.

- The term $-\left(\tau - \tau\min\left(\alpha, c\bar{f}\right)\right)\Delta$ captures the tax burden incurred this period.
- The term $-c\bar{f}\Delta$ represents the instantaneous coupon payment on the existing debt level $\bar{f}$.
- The term $\frac{\alpha}{c} - \bar{f}$ reflects the payment associated with adjusting the debt level.
- Finally, $V_e^+\left(\frac{\alpha}{c}\right)$ denotes the continuation value of the firm at the end of the period.

Since $\bar{f}$ is the threshold at which the firm is indifferent between defaulting and continuing, this value must equal zero:

$$Q_2(\bar{f}, c) = -\left(\tau - \tau \min\left(\alpha, c\bar{f}\right) + c\bar{f}\right)\Delta + \frac{\alpha}{c} - \bar{f} + V_e^+\left(\frac{\alpha}{c}\right) = 0. \tag{19}$$

This gives us the second supporting condition.

Determination of $V_e^+(\cdot)$: Notice that Equation (19) crucially depends on $V_e^+\left(\frac{\alpha}{c}\right)$, which is endogenously determined in equilibrium. Specifically, by Equations (5) and (9), for any f_t^+, we can write $V_e^+(f_t^+)$ in the following recursive forms:

$$\begin{aligned}
&V_e^+(f_t^+) = \exp\left(-r\Delta\right)\Delta \int_{-\infty}^{\infty} g(Z_{t+1})\phi(Z_{t+1})dZ_{t+1} \\
&+ \exp\left(-r\Delta\right)\Delta \int_{Z(f_t^+)}^{\infty} \left(-\tau g(Z_{t+1}) + \tau \min\{cf_t^+, \alpha g(Z_{t+1})\} - cf_t^+\right)\phi(Z_{t+1})dZ_{t+1} \\
&+ \exp\left(-r\Delta\right)\left[-f_t^+ \int_{Z(f_t^+)}^{\infty} \phi(Z_{t+1})dZ_{t+1} + \left(\frac{\alpha}{c} + V_e\left(\frac{\alpha}{c}\right)\right)\int_{Z(f_t^+)}^{\infty} g(Z_{t+1})\phi(Z_{t+1})dZ_{t+1}\right],
\end{aligned} \tag{20}$$

where $\phi(\cdot)$ represents the density function of a standard normal distribution. To interpret the equation, the first line represents the firm's pre-tax income for the next period, the second line represents the firm's tax and coupon payments, and the third line represents the firm's payoff from debt level adjustment and its future continuation value.

Let us define

$$Z_c(f_t^+) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\left(\frac{c}{\alpha}f_t^+\right) - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right)$$

to represent the cash flow shock such that $cf_t^+ = \alpha g(Z_{t+1})$, and

$$\tilde{Z}(f_t^+) = \max\{Z_c(f_t^+), Z(f_t^+)\}$$

to better capture the tax shield calculation.

Therefore, the firm is going to default if it experiences a negative cash flow $Z_{t+1} \leq Z(f_t^+)$; it collects a tax shield $\alpha g(Z_{t+1})$ if $Z_{t+1} \in \left(Z(f_t^+), \tilde{Z}(f_t^+)\right]$; and it collects a tax shield τf_t^+ if $Z_{t+1} > \tilde{Z}(f_t^+)$.

By properties of the normal distribution, we have:

$$\begin{aligned}V_e^+(f_t^+) = \exp(-r\Delta) \Bigg[&\Delta \exp(\mu\Delta) - \tau\Delta\exp(\mu\Delta)\left[1-\Phi\left(Z(f_t^+)-\sigma\sqrt{\Delta}\right)\right] \\ &+ \tau\alpha\Delta\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}(f_t^+)-\sigma\sqrt{\Delta}\right)-\Phi\left(Z(f_t^+)-\sigma\sqrt{\Delta}\right)\right] \\ &+ \tau c f_t^+\Delta\left[1-\Phi\left(\tilde{Z}(f_t^+)\right)\right] - \left(cf_t^+\Delta + f_t^+\right)\left[1-\Phi\left(Z(f_t^+)\right)\right] \\ &+ \left[\frac{\alpha}{c} + V_e^+\left(\frac{\alpha}{c}\right)\right]\exp(\mu\Delta)\left[1-\Phi\left(Z(f_t^+)-\sigma\sqrt{\Delta}\right)\right]\Bigg]. \end{aligned} \tag{21}$$

To interpret each term in Equation (21):

- $\Delta\exp(\mu\Delta)$ represents the expected cash flow next period.
- $-\tau\Delta\exp(\mu\Delta)\left[1-\Phi\left(Z(f_t^+)-\sigma\sqrt{\Delta}\right)\right]$ represents the expected tax payment (without tax shield).
- $\tau\alpha\Delta\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}(f_t^+)-\sigma\sqrt{\Delta}\right)-\Phi\left(Z(f_t^+)-\sigma\sqrt{\Delta}\right)\right]+\tau c f_t^+\Delta\left[1-\Phi\left(\tilde{Z}(f_t^+)\right)\right]$ represents the expected tax shield.
- $-cf_t^+\Delta\left[1-\Phi\left(Z(f_t^+)\right)\right]$ represents the expected coupon payment.
- $-f_t^+\left[1-\Phi\left(Z(f_t^+)\right)\right] + \frac{\alpha}{c}\exp(\mu\Delta)\left[1-\Phi\left(Z(f_t^+)-\sigma\sqrt{\Delta}\right)\right]$ represents the expected payoff from debt level changes (maturing, repurchase, or issuance).
- $V_e^+\left(\frac{\alpha}{c}\right)\exp(\mu\Delta)\left[1-\Phi\left(Z(f_t^+)-\sigma\sqrt{\Delta}\right)\right]$ represents the expected continuation value.

Plugging $f_t^+ = \frac{\alpha}{c}$ into Equation (21), we can derive $V_e^+\left(\frac{\alpha}{c}\right)$:

$$\begin{aligned}V_e^+\left(\frac{\alpha}{c}\right) = &\frac{\exp(-r\Delta)}{1-\exp\left(-(r-\mu)\Delta\right)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]} \\ &\Bigg[\Delta\exp(\mu\Delta) - \tau\Delta\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right] \\ &+ \tau\alpha\Delta\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right] \\ &+ \tau\alpha\Delta\left[1-\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)\right)\right] - \left(\alpha\Delta + \frac{\alpha}{c}\right)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right] \\ &+ \frac{\alpha}{c}\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]\Bigg]. \end{aligned} \tag{22}$$

In addition, we derive the value of $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right)$, $\lim_{\bar{f}\to 0} V_e^+\left(\frac{\alpha}{c}\right)$, and $\lim_{\bar{f}\to\infty} V_e^+\left(\frac{\alpha}{c}\right)$ in Lemma 1. These results will be useful in the following analysis. To simplify the expressions, we define the following notations:

$$V_e^{+\Delta\to 0}(c) := \frac{1}{r-\mu}\left(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{c}\mu\right),$$

$$V_e^{+\bar{f}\to 0}(c) := \exp(-(r-\mu)\Delta)\Delta,$$

and

$$V_e^{+\bar{f}\to\infty}(c) := \frac{\exp(-r\Delta)}{1-\exp(-(r-\mu)\Delta)}\Big[\exp(\mu\Delta)\,\Delta-\tau(1-\alpha)\exp(\mu\Delta)\Delta$$
$$-\frac{\alpha}{c}(1+c\Delta)+\frac{\alpha}{c}\exp(\mu\Delta)\Big].$$

Lemma 1. *For any $\bar{f} > \frac{\alpha}{c}$, $\lim_{\Delta\to 0} V_e^{+}\left(\frac{\alpha}{c}\right) = V_e^{+\Delta\to 0}(c)$. In addition, $\lim_{\bar{f}\to 0} V_e^{+}\left(\frac{\alpha}{c}\right) = V_e^{+\bar{f}\to 0}(c)$, which is positive, and $\lim_{\bar{f}\to\infty} V_e^{+}\left(\frac{\alpha}{c}\right) = V_e^{+\bar{f}\to\infty}(c)$, which is bounded.*

Existence of $(\bar{f}, c)$ Satisfying Two Supporting Conditions

Now, we need to establish that there exist $(\bar{f}, c)$ such that two supporting conditions, Equations (17) and (19), hold.

First, we show that for any c satisfying Assumption 1, there exists a function $\bar{f}^{Q_1}(c)$ such that $(\bar{f}^{Q_1}(c), c)$ satisfies Equation (17). The intuition is that, given Assumption 1, the firm receives a coupon payment larger than the discounting factor, making the present value of the debt strictly greater than the face value in the absence of default risk. In this scenario, if the default boundary $\bar{f}$ is very high, the firm is highly unlikely to default, approximating a no-default situation and resulting in a higher expected present value than the face value. Conversely, if the default boundary $\bar{f}$ is very low, the firm is highly likely to default, rendering the debt nearly worthless and leading to an expected present value significantly below the face value. To satisfy Equation (17), there exists an intermediate $\bar{f}$ representing a moderate default risk, ensuring that the expected present value precisely equals the face value. Given the continuity of $Q_1(\bar{f}, c)$, $\bar{f}^{Q_1}(c)$ can be constructed as continuous.

Second, when Δ is not large, we show that for any c, there exists a function $\bar{f}^{Q_2}(c)$ such that $(\bar{f}^{Q_2}(c), c)$ satisfies Equation (19). In this case, when the debt level is low, the firm has a manageable debt burden, resulting in a continuation value that exceeds its liabilities, including both taxes and debt, thus dissuading default. Conversely, when the debt level is high, the firm faces a significant debt burden, which incentivizes default. Consequently, there exists a medium level of debt burden $\bar{f}^{Q_2}(c)$ at which the firm is indifferent between defaulting and continuing operations. Given the continuity of $Q_2(\bar{f}, c)$, $\bar{f}^{Q_2}(c)$ can also be constructed as continuous.

Therefore, as long as we can have a $c > \bar{c}$ such that $\bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$, this c and $\bar{f} :=$

$\bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$ will satisfy both supporting conditions. We show that such a c exists when Δ is close to zero. The intuition is as follows:

First, when $\Delta \to 0$, $\bar{f}^{Q_2}(c) \to \frac{\alpha}{c} + V_e^{+\Delta\to 0}(c)$. This is because the firm needs to retire $\bar{f}^{Q_2}(c) - \frac{\alpha}{c}$ amount of debt and gains a continuation value converging to $V_e^{+\Delta\to 0}(c)$. The instantaneous payments of tax and coupon are negligible since each time period is very short. Consequently, the firm breaks even at $\bar{f}^{Q_2}(c) \to \frac{\alpha}{c} + V_e^{+\Delta\to 0}(c) > \frac{\alpha}{c}$.

Second, given a small Δ, $\bar{f}^{Q_1}(c)$ rapidly decreases from infinity to $\frac{\alpha}{c}$ as c increases from $\bar{c}$. This is because, when c slightly exceeds $\bar{c}$, the firm must have a significant probability of default for Equation (17) to hold. In this case, with Δ being very small and the probability of any large shock approaching zero, the default boundary $\bar{f}^{Q_1}(c)$ should approximate $\frac{\alpha}{c}$ to ensure the firm defaults with a significant probability. On the other hand, when c is nearly equal to $\bar{c}$, making the present value of debt almost equal to the face value in the absence of default, the default boundary $\bar{f}^{Q_1}(c)$ must be very large to ensure minimal default risk and uphold Equation (17).

Given that $\bar{f}^{Q_2}(c)$ converges to $\frac{\alpha}{c} + V_e^{+\Delta\to 0}(c)$ and $\bar{f}^{Q_1}(c)$ rapidly decreases from infinity to $\frac{\alpha}{c}$ when $c > \bar{c}$, $\bar{f}^{Q_1}(c)$ must intersect with $\bar{f}^{Q_2}(c)$ for some $c > \bar{c}$. This implies that we can find $(\bar{f}, c)$ satisfying both Equations (17) and (19) when Δ is small.

The above intuition is illustrated in Figure 1. In this figure, the blue line represents $\bar{f}^{Q_1}(c)$, while the green line represents $\bar{f}^{Q_2}(c)$. The blue line decreases rapidly from infinity, whereas the green line decreases much more gradually. Consequently, they intersect at the point $(26.143, 0.05143)$.

Lemma 2. *We can find $\bar{\Delta}$ such that for any $\Delta < \bar{\Delta}$, there exists $(\bar{f}, c)$ such that the two supporting conditions, Equations* (17) *and* (19)*, hold.*

3.3 Equilibrium Verification

In this section, we are going to verify that as long as $(\bar{f}, c)$ satisfy the supporting conditions, the conjectured strategies indeed constitute an equilibrium. We establish this result by verifying the equilibrium conditions in each period through backward induction.

First, the firm's conjectured default strategy in Equation (2a) satisfies the optimality condition given in Equation (12). This follows directly from the conjectured form: the firm will not default if and only if its required payment is smaller than its continuation value, thereby confirming the strategy's optimality.

Second, the debt-holders' strategy satisfies the optimality condition given in Equation (13), as implied by the debt price function in Equation (16) and the supporting condition in Equation (17). Specifically, Equation (16) shows that the debt price is decreasing in f_t^+, since a higher debt ratio increases the likelihood of default in the next period. Together

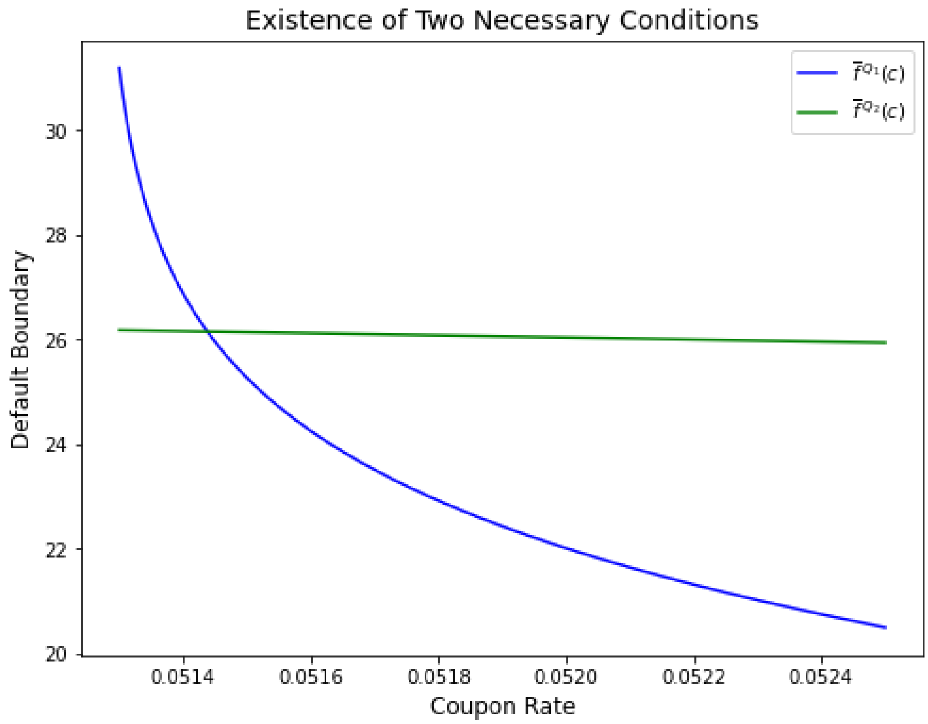

Figure 1: This figure demonstrates that there exists a pair $(\bar{f}, c)$ satisfying Equations (17) and (19). The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, $\alpha = 0.3$, and $\Delta = 1$.

with Equation (17), it follows that if $f_t^+ > \frac{\alpha}{c}$, the debt price is less than 1; if $f_t^+ < \frac{\alpha}{c}$, the debt price exceeds 1; and if $f_t^+ = \frac{\alpha}{c}$, the debt price is exactly 1. When the post-adjustment debt ratio $f_{\gamma_t}^+ \leq \frac{\alpha}{c}$, debt-holders anticipate that the firm will not default. In this case, the final debt ratio at the end of the period remains $f_t^+ = f_{\gamma_t}^+$, since the demand clause is not exercised according to the equilibrium strategy. Consequently, the debt price satisfies $P(f_t^+) = P(f_{\gamma_t}^+) \geq P\left(\frac{\alpha}{c}\right) = 1$, which justifies the debt-holders' decision not to exercise the demand clause and instead hold the debt to receive its full value. In contrast, when $f_{\gamma_t}^+ > \frac{\alpha}{c}$, debt-holders understand that, under the equilibrium strategy, the effective debt ratio at the end of the period—after the exercise of the demand clause—will be reduced to $\frac{\alpha}{c}$. If the firm does not default, the debt price equals $P\left(\frac{\alpha}{c}\right) = 1$, matching the face value. If the firm defaults, the debt-holders receive nothing regardless of whether they hold or redeem the debt. Therefore, they are indifferent between exercising the demand clause or not, justifying the conjectured mixed strategy.

Third, we verify that the firm's debt issuance strategy (2b) satisfies the optimality condition stated in Equation (11).

To begin, consider the *non-default region*, where $f_t^- < \bar{f}$. In this case, the firm is expected to maintain a debt level equal to $\frac{\alpha}{c}$. We first show that the firm cannot profitably deviate by *underissuing* or *overrepurchasing* debt—that is, choosing

$$\gamma_t < \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$$

is not profitable. Specifically, such a deviation is *dominated* by the equilibrium strategy

$$\gamma(f_t^-) = \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$$

in the limit as $\Delta \to 0$. The intuition is as follows: choosing a debt level that is too low reduces the firm's tax-shield benefits. Meanwhile, as $\Delta \to 0$, the probability of a sufficiently large shock that would trigger bankruptcy becomes negligible. Therefore, the gain from reducing expected bankruptcy costs by having less debt is minimal, while the loss in tax benefits becomes relatively more significant. As a result, having too little debt—that is, choosing γ_t strictly below the conjectured strategy—is suboptimal. We formalize this argument in Lemma 3.

Lemma 3. *When $f_t^- < \bar{f}$, the firm cannot benefit from a deviation where $\gamma_t < \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$ as $\Delta \to 0$.*

Meanwhile, *overissuing* or *underrepurchasing* debt in this case is also suboptimal. That is, choosing

$$\gamma_t > \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$$

is not profitable. In this case, the debt-holders will exercise the demandable clause and put the debt back to the firm. If the firm does not default after this deviation, the demandable clause effectively un-levers the excessive leverage, rendering the deviation irrelevant. If the firm does default following the deviation, it receives nothing, since the debt-holders—rationally anticipating default—will not pay for the newly issued debt, and the continuation value of the firm drops to zero. Since the firm's value was positive prior to the deviation, such a deviation is not profitable.

Now, let us consider the *default region*, where $f_t^- \geq \bar{f}$. In this region, the firm will default and receive zero under its equilibrium strategy. If the firm does not default after the deviation, debt-holders, rationally anticipating non-default, will price the debt accordingly, implying a high debt price. Given the debt-holders' strategy, the firm must adjust the debt level from f_t^- to weakly below $\frac{\alpha}{c}$ following the deviation—either through debt repurchase, debt maturing, or put-back from the demandable clause—all of which are costly. Together with the definition of the default boundary, the firm receives a negative payoff from such an action. Therefore, this deviation is not profitable. If the firm still defaults after the deviation, then as discussed in the previous paragraph, the firm again receives zero from the deviation.

Thus, this deviation is also not profitable. We formalize this argument in Lemma 4.

Lemma 4. *As $\Delta \to 0$, the firm cannot profit from any deviation γ_t from the equilibrium issuance strategy.*

Since all the proposed strategies are shown to be optimal, we establish that they collectively form an equilibrium.

Proposition 1. *The strategies outlined in Section 3.1 constitute an equilibrium when $\Delta \to 0$.*

4 Discussion

The demand clause serves as an important tool to discipline the firm from accumulating excessive debt. The intuition is that if the firm holds an excessive amount of debt exceeding the targeted debt level, the debt-holders will immediately exercise the demand clause, forcing the firm to repay and subsequently retire the excess debt. This mechanism helps break the leverage ratchet effect, where the debt ratio continues to increase. There are two important issues to discuss. First, whether the demand clause requires strong conditions to implement, challenging the notion that the firm has no dynamic commitment power. Second, whether the demand clause enhances the firm's welfare, making the firm willing to adopt this clause from the outset.

4.1 Implementation Conditions

To effectively implement the demand clause, two key conditions must be met. First, debt-holders need to observe the firm's debt ratio. Second, debt-holders must have the ability to request repayment, and if the firm refuses, it must face bankruptcy. Both of these implementation conditions are widely assumed in the literature on debt pricing.

The first condition, that debt-holders can observe the debt ratio, is necessary for the market to competitively price the debt based on this ratio. If debt-holders cannot observe the debt ratio, they can no longer base their bids on it, and there would be no market forces to ensure the debt price reflects its true present value.

The second condition, allowing debt-holders to request repayment and impose bankruptcy if denied, parallels the situation where debt-holders can demand coupon and principal payments when the debt matures, leading to bankruptcy if the request is rejected. This condition is essential for the debt to have any value, as without it, the firm could avoid repayment without consequence, making it difficult to incentivize repayment of any debt.

Thus, the demand clause can be implemented as long as we have a well-functioning debt market where the debt is fairly priced, and the firm is disciplined by the threat of bankruptcy, both of which are commonly assumed in the literature.

From a practical perspective, demand clauses are widely used in reality. For example, depositors can withdraw their deposits at any time. They are likely to do so when they observe negative news about the bank, and the bank does not manage the situation properly, a scenario predicted by our equilibrium (as an off-equilibrium path action).

4.2 Welfare Analysis

As argued in the Leverage Ratchet effect, the firm loses a significant amount of equity value when it cannot restrain itself from issuing more debt. The intuition is that the firm incurs excessive bankruptcy costs as it continues to increase its leverage. For example, DeMarzo and He (2021) argues that equity holders do not benefit from issuing additional debt. The equity value without any commitment power is essentially the same as if the firm could not issue any debt.

In contrast, in the equilibrium we have constructed with the demand clause, we show that the equity value asymptotically achieves the equity value as if the firm had full commitment power when $\Delta \to 0$.

4.2.1 Asymptotic Equity Value with Full Commitment Power

Given any $\Delta < \bar{\Delta}$, let us denote $(\bar{f}, c)$ satisfy two necessary conditions as $(\bar{f}(\Delta), c(\Delta))$. By our construction of $\bar{f}^{Q_2}(c)$, we have $\bar{f}(\Delta) \to \frac{\alpha}{c(\Delta)} + V_e^{+\Delta\to 0}(c(\Delta))$, when $\Delta \to 0$. This implies that $\bar{f}(\Delta) > \frac{\alpha}{c(\Delta)}$. By Equation (17), we have $c(\Delta) \to r$ when $\Delta \to 0$. Figure 2 illustrates the convergence of $c(\Delta)$ to r as Δ approaches zero.

Lemma 5. *We have* $\lim_{\Delta\to 0} c(\Delta) = r$ *and* $\lim_{\Delta\to 0} \bar{f}(\Delta) = \frac{\alpha}{r} + V_e^{+\Delta\to 0}(r)$.

As a result, the equity value given $f_t^+ = \frac{\alpha}{c(\Delta)} \to \frac{\alpha}{r}$ can be calculated as:

$$\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c(\Delta)}\right) = \frac{1-(1-\alpha)\tau}{r-\mu} - \frac{\alpha}{r}. \tag{23}$$

Proposition 2. *As* $\Delta \to 0$*, the firm value with demandable debt converges to* $\frac{1-(1-\alpha)\tau}{r-\mu}$*, which represents the firm value with full tax shield benefits and zero bankruptcy costs in the limit. More specifically, the equity holders receive* $\frac{\alpha}{r}$ *in cash from the demandable debt issuance, and*

$$\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c(\Delta)}\right) = \frac{1-(1-\alpha)\tau}{r-\mu} - \frac{\alpha}{r}$$

as the remaining equity.

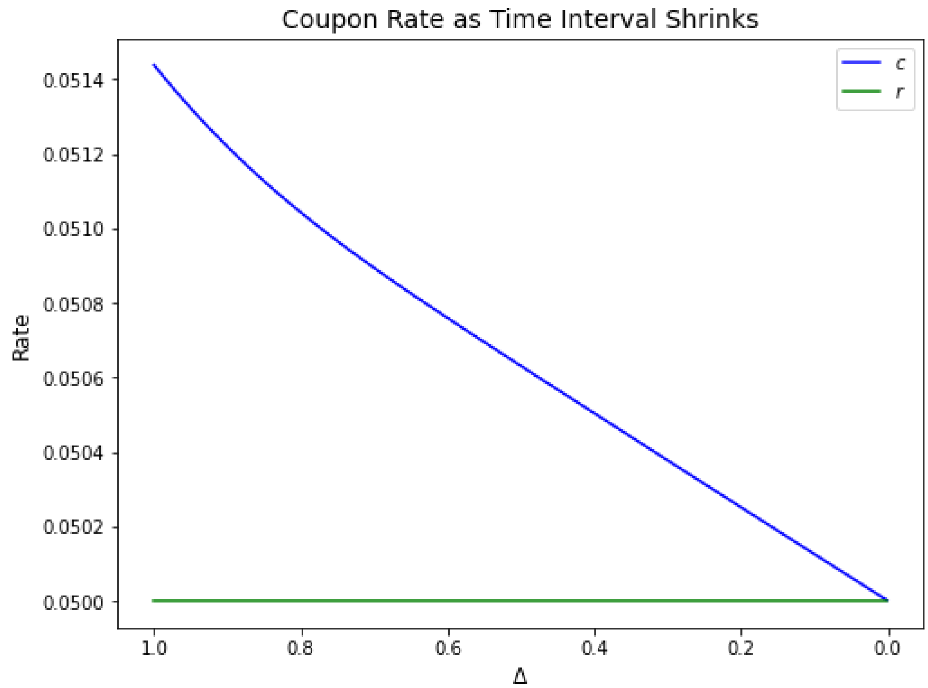

Figure 2: This figure demonstrates how $c(\Delta)$ converges to r when Δ converges to 0. The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, and $\alpha = 0.3$.

Notice that $\frac{1-(1-\alpha)\tau}{r-\mu}$ represents the present value of all future after-tax income along with the tax shield per unit of current cash flow, and $\frac{\alpha}{r}$ represents the value of debt per unit of current cash flow. Because the firm can adjust its debt level frequently and adheres to the targeted debt ratio $\frac{\alpha}{c} \to \frac{\alpha}{r}$, it can fully capture the tax shield and asymptotically avoid any possibility of default. Figure 3 depicts how the default probability converges to zero when Δ converges to 0.

Given that the initial amount of debt is issued at par value, according to our first necessary condition, Equation (17), the equity holders receive $\frac{1-(1-\alpha)\tau}{r-\mu}X_0$ from this firm, which equals the equity value as if the firm could commit to maintaining the debt level, fully achieving the tax shield, and never defaulting. Figure 4 illustrates how the firm value with demandable debt converges to the firm value as if the firm possesses full commitment power when Δ approaches zero. The blue line represents the firm's value under our mechanism with demandable debt. In contrast, the green line illustrates the firm's value assuming full commitment power, enabling the firm to fully capture the tax shield benefits while incurring zero bankruptcy costs, as derived in Proposition 2.

Since the equity value is higher with the demand clause, the firm will optimally adopt the demand clause from the outset.

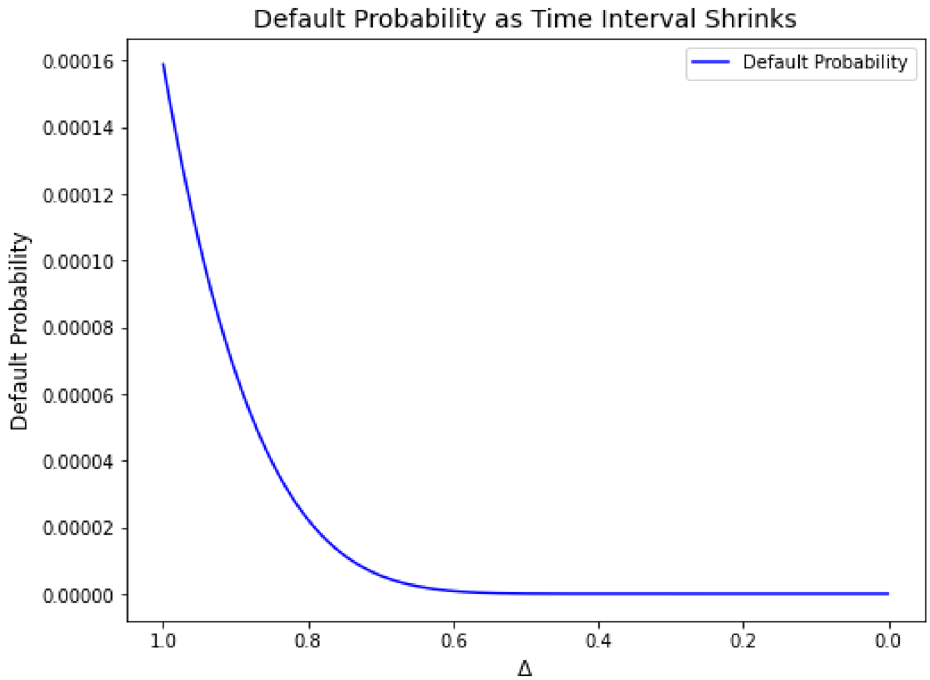

Figure 3: This figure demonstrates how the default probability converges to zero when Δ converges to 0. The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, and $\alpha = 0.3$.

4.2.2 Frequent Issuance of Debt

In the literature, the ability to frequently issue debt typically leads to a significant commitment problem and destroys firm value. As a result, if the firm can trade more frequently, it tends to lose more value, similar to the implications of the Coase Conjecture. However, this is not the case in our model.

In our case, the firm can credibly adjust its leverage ratio to the targeted debt ratio $\frac{\alpha}{c}$ provided the debt ratio remains below the default threshold. Thus, frequent trading actually works in the firm's favor. Specifically, if the firm can trade very frequently, it can immediately respond to any cash flow shocks before those shocks accumulate. That is, when the firm experiences a positive cash flow shock, it can increase its leverage to maximize the tax shield benefit. Conversely, if the firm experiences a negative cash flow shock, it can immediately decrease its leverage before the negative shock becomes too large for the firm to handle comfortably. In the limit of continuous trading, the bankruptcy probability converges to zero. Consequently, the firm can fully capture tax shield advantages without incurring default costs, achieving first-best efficiency.

This idea that the frequency of trading enhances efficiency is also discussed in the literature on market completeness, such as Kreps (1982) and Duffie and Huang (1985).

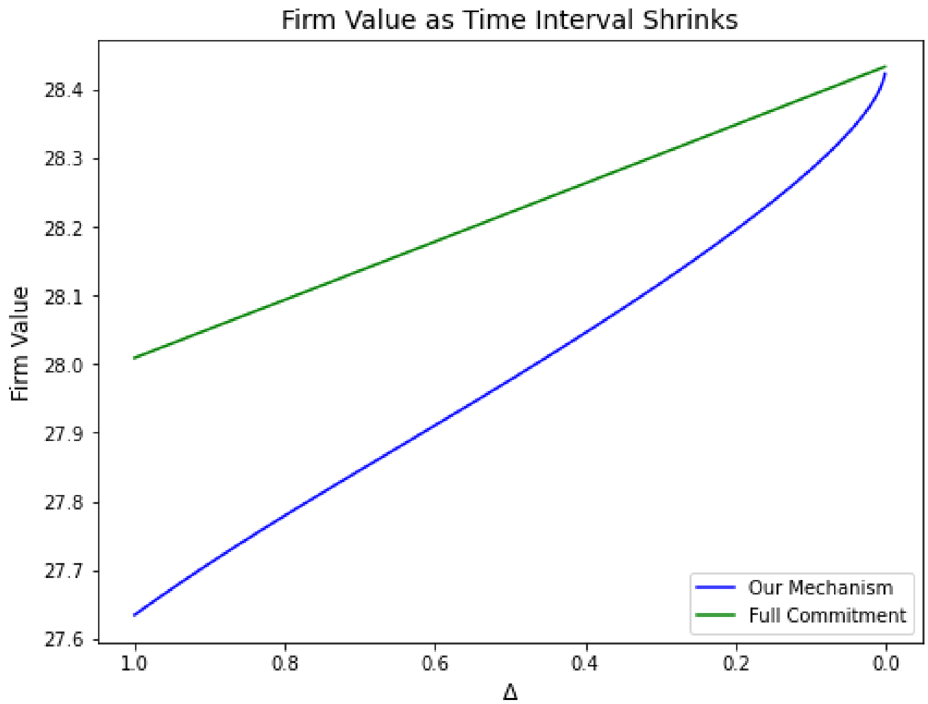

Figure 4: This figure demonstrates how the firm value with demandable debt converges to the firm value as if the firm has the full commitment power when Δ converges to 0. The parameters used are: $\mu = 0.02$, $\sigma = 0.4$, $r = 0.05$, $\tau = 0.21$, and $\alpha = 0.3$, and $X_0 = 1$.

5 Conclusion

This paper has shown that demandable debt is an effective mechanism for addressing the leverage ratchet effect. By incorporating a demand clause, debt-holders gain the ability to discipline the firm, ensuring that the debt ratio remains within targeted levels and default risks are minimized. This approach enables firms to secure tax shield benefits while reducing the likelihood of excessive bankruptcy costs, thereby approximating the optimal outcomes obtained in full commitment scenarios.

Moreover, our findings indicate that demandable debt can be implemented under conditions that are already standard in the existing literature. This insight suggests a potential evolution in how firms and regulators approach the structure of debt. Given that our results demonstrate the potential for firms to enhance their welfare through the adoption of demandable debt, such a shift could be driven by market forces.

Future research should focus on empirical evaluations of demandable debt's effectiveness. Additionally, examining variations of the model across different economic contexts would help determine its broader applicability and potential impact.

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6 Proofs

6.1 Math Preparation

In this paper, we focus on normally distributed shocks, which are widely adopted in the literature. These shocks have very nice analytical properties that serve as an important tool to smooth our analysis. Additionally, they converge to a geometric Brownian motion as the length of each period, Δ, converges to zero. This implies that it becomes increasingly unlikely to experience a large shock within one period as Δ shrinks, providing the firm an opportunity to adjust its debt level and absorb the shock. As a result, we can asymptotically achieve the equity value as if the firm had full commitment power.

By the property of normal distribution, for any $x > 0$, we have $1-\Phi(x) \in \left((1-\frac{1}{x^2})\frac{\phi(x)}{x}, \frac{\phi(x)}{x}\right)$. As a result, for any $x \to \infty$, we could write

$$1 - \Phi(x) = \frac{\phi(x)}{x}\left(1 + O(\frac{1}{x^2})\right), \tag{24}$$

where the notation $O(\frac{1}{x^2})$ refers to an order of magnitude in terms of the variable $\frac{1}{x^2}$. Similarly, for any $x \to -\infty$, we could write

$$\Phi(x) = -\frac{\phi(x)}{x}\left(1 + O(\frac{1}{x^2})\right). \tag{25}$$

Let us consider the case where $f_1 > f_2$. When $\Delta \to 0$, we have

$$Z\left(f_1, f_2\right) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{f_2}{f_1} - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{f_2}{f_1}\right) - \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{\Delta} \to -\infty \tag{26}$$

In addition, we have

$$\lim_{\Delta\to 0}\frac{\phi(Z\left(f_1, f_2\right))}{Z\left(f_1, f_2\right)\Delta} = \lim_{\Delta\to 0}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Z\left(f_1, f_2\right)^2}{2}\right)\frac{1}{Z\left(f_1, f_2\right)\Delta} = 0. \tag{27}$$

As a result, we have $\lim_{\Delta\to 0}\frac{\Phi(Z(f_1,f_2))}{\Delta} = 0$. By the same logic, we have for any constant A, $\lim_{\Delta\to 0}\frac{\Phi\left(Z(f_1,f_2)+A\sqrt{\Delta}\right)}{\Delta} = 0$.

In addition, if there exists any $C > 0$ such that

$$\frac{\alpha}{c} < \frac{\bar{f}}{1+C},$$

we have $\lim_{\Delta\to 0}\frac{\Phi(Z(f_1,f_2))}{\Delta}$ converges to 0 uniformly. This is true for $\lim_{\Delta\to 0}\frac{\Phi\left(Z(f_1,f_2)+A\sqrt{\Delta}\right)}{\Delta}$

as well.

With a slight abuse of notation, we define

$$\Phi(f_2) := \Phi\left(Z(f_1, f_2)\right),$$

and

$$\phi(f_2) := \phi\left(Z(f_1, f_2)\right),$$

The derivative of $\Phi(f_2)$ with respect to f_2 is given by

$$\frac{\partial \Phi(f_2)}{\partial f_2} = \phi(f_2) \cdot \frac{1}{\sigma\sqrt{\Delta}} \cdot \frac{1}{f_2},$$

and the derivative of $\frac{\Phi(f_2)}{\Delta}$ is

$$\frac{\partial}{\partial f_2}\left(\frac{\Phi(f_2)}{\Delta}\right) = \phi(f_2) \cdot \frac{1}{\sigma\Delta^{3/2}} \cdot \frac{1}{f_2}.$$

Therefore, we have

$$\lim_{\Delta\to 0} \frac{\partial \Phi(f_2)}{\partial f_2} = 0, \qquad \lim_{\Delta\to 0} \frac{\partial}{\partial f_2}\left(\frac{\Phi(f_2)}{\Delta}\right) = 0.$$

For any $f_1 > f_2 > f_2'$, by Rolle's Theorem, there exists some $f_2'' \in [f_2', f_2]$ such that

$$\Phi(f_2) - \Phi(f_2') = \frac{\partial \Phi(f_2'')}{\partial f_2''}(f_2 - f_2'), \quad \frac{\Phi(f_2)}{\Delta} - \frac{\Phi(f_2')}{\Delta} = \frac{\partial}{\partial f_2''}\left(\frac{\Phi(f_2'')}{\Delta}\right)(f_2 - f_2')$$

Hence, we conclude that

$$\frac{\Phi(f_2) - \Phi(f_2')}{f_2 - f_2'} \to 0, \quad \frac{\frac{\Phi(f_2)}{\Delta} - \frac{\Phi(f_2')}{\Delta}}{f_2 - f_2'} \to 0,$$

uniformly as $\Delta \to 0$.

Similarly, for any fixed constant A, we have

$$\frac{\Phi\left(Z(f_1, f_2) + A\sqrt{\Delta}\right) - \Phi\left(Z(f_1, f_2') + A\sqrt{\Delta}\right)}{f_2 - f_2'} \to 0,$$
$$\frac{\frac{\Phi\left(Z(f_1,f_2)+A\sqrt{\Delta}\right)}{\Delta} - \frac{\Phi\left(Z(f_1,f_2')+A\sqrt{\Delta}\right)}{\Delta}}{f_2 - f_2'} \to 0,$$

uniformly as $\Delta \to 0$.

6.2 Proof of Lemma 1

Let us first derive $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right)$ given any c and $\bar{f}$ such that $\bar{f} > \frac{\alpha}{c}$. Let us decompose $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right)$ into several different terms. The first term is

$$\lim_{\Delta\to 0} \frac{\exp(-r\Delta)}{1-\exp(-(r-\mu)\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]}\Delta = \frac{1}{r-\mu}, \tag{28}$$

which resembles the cap rate in a perpetuity formula.

The second term is

$$\begin{aligned}
&\lim_{\Delta\to 0}\Bigg(\exp(\mu\Delta) - \tau\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right] \\
&+\tau\alpha\exp(\mu\Delta)\left[\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right] \\
&+\tau\alpha\left[1-\Phi\left(\tilde{Z}\left(\frac{\alpha}{c}\right)\right)\right]-\alpha\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right]\Bigg) \\
&= 1-\tau(1-\alpha)-\alpha,
\end{aligned} \tag{29}$$

which represents the after-tax income and coupon payment for the initial outstanding debt.

The last term is

$$\begin{aligned}
&\lim_{\Delta\to 0}\frac{1}{\Delta}\left(-\frac{\alpha}{c}\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right]+\frac{\alpha}{c}\exp(\mu\Delta)\left[1-\Phi\left(Z\left(\frac{\alpha}{c}\right)-\sigma\sqrt{\Delta}\right)\right]\right) \\
&=\lim_{\Delta\to 0}\frac{1}{\Delta}\left(-\frac{\alpha}{c}+\frac{\alpha}{c}\exp(\mu\Delta)\right)=\frac{\alpha}{c}\mu,
\end{aligned} \tag{30}$$

which represents the value collected from issuing debt because the firm grows. In addition, all three terms converge uniformly when $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$.

As a result, we have $\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right) = V_e^{+\Delta\to 0}(c) := \frac{1}{r-\mu}\left(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{c}\mu\right)$. The convergence is uniform for $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$.

The derivations for $\lim_{\bar{f}\to 0} V_e^+\left(\frac{\alpha}{c}\right)$, and $\lim_{\bar{f}\to\infty} V_e^+\left(\frac{\alpha}{c}\right)$ can be directly completed from the given functions. The steps are straightforward and are left as an exercise to the reader to verify.

6.3 Proof of Lemma 2

In this section, we prove that there exist $(\bar{f}, c)$ such that two supporting conditions, Equations (17) and (19), hold.

First, $Q_1(\bar{f}, c)$ is increasing in both $\bar{f}$ and c. The intuition is twofold: 1) with a larger $\bar{f}$, the firm will only default after experiencing a more significant negative shock, reducing the likelihood of default and increasing the probability that debt-holders will be paid; and 2) with a larger c, debt-holders receive more coupon payments when the firm does not default. Furthermore, for any c satisfying Assumption 1, we have

$$\lim_{\bar{f}\to\infty} Q_1(\bar{f},c) = \exp(-r\Delta)(c\Delta+1) > 1, \quad \lim_{\bar{f}\to 0} Q_1(\bar{f},c) = 0.$$

That is, if $\bar{f}$ is very large, the firm is unlikely to go bankrupt, and under Assumption 1, the present value of the debt exceeds its face value. Conversely, if $\bar{f}$ approaches zero, the firm will almost surely go bankrupt, and the present value of the debt approaches zero. By the continuity of $Q_1(\bar{f},c)$, for any c satisfying Assumption 1, there always exists a $\bar{f}$ that satisfies Equation (17).

As a result, for any c satisfying Assumption 1, there exists a function $\bar{f}^{Q_1}(c)$ such that Equation (17) holds for the pair $(\bar{f}^{Q_1}(c), c)$. Furthermore, $\bar{f}^{Q_1}(c)$ is a decreasing and continuous function of c. Intuitively, to maintain the debt price at 1 when the debt ratio is $\frac{\alpha}{c}$, the firm must be more prone to default as the coupon payment c increases. This ensures that the expected present value of the debt remains equal to 1.

Second, by Lemma 1, we have

$$\lim_{\bar{f}\to 0} Q_2(\bar{f},c) = -\tau\Delta + \frac{\alpha}{c} + \exp(-(r-\mu)\Delta)\Delta, \tag{31}$$

which is greater than 0 given any $\Delta \leq \bar{\Delta}'$ for some $\bar{\Delta}'$. Intuitively, when $\bar{f}$ is too small, implying a small debt burden, the firm should strictly prefer to honor its various liabilities given $\bar{f}$.

By Lemma 1, we have $\lim_{\bar{f}\to\infty} V_e^+\left(\frac{\alpha}{c}\right)$ bounded. As a result,

$$\lim_{\bar{f}\to\infty} Q_2(\bar{f},c) \to -\infty. \tag{32}$$

Intuitively, when $\bar{f}$ is too large, implying a large debt burden, the firm should strictly prefer to default.

By continuity of $Q_2(\bar{f},c)$, for any c, we can define a function $\bar{f}^{Q_2}(c)$ such that $(\bar{f}^{Q_2}(c), c)$ solves Equation (19). That is, there exists a medium level $\bar{f}$ such that the firm is indifferent between honoring its liabilities and defaulting. In addition, $\bar{f}^{Q_2}(c)$ can be continuous in c given the continuity of $Q_2(\bar{f},c)$.

Now, we will show that $Q_2(\bar{f},c)$ can be constructed to converge to the value $\frac{\alpha}{c} + V_e^{+\Delta\to 0}(c)$. To start, for any $\bar{f}$ and c such that $\bar{f} > \frac{\alpha}{c}$, we have the limiting equity value:

$$\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c}\right) = V_e^{+\Delta\to 0}(c) := \frac{1}{r-\mu}\left[(1-\alpha)(1-\tau) + \frac{\alpha}{c}\mu\right] > 0.$$

This convergence is uniform for $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$. As a result,

$$\lim_{\Delta \to 0} Q_2(\bar{f}, c) = \frac{\alpha}{c} - \bar{f} + V_e^{+\Delta \to 0}(c),$$

and this convergence is uniform for $c \geq r$ and $\frac{\alpha}{c} < \frac{\bar{f}}{1+C}$ with some $C > 0$.

For any $c \geq r$ and $\epsilon \in (0, \frac{1}{2})$, let us consider $\bar{f}' = \frac{\alpha}{c} + (1-\epsilon)V_e^{+\Delta \to 0}(c)$ and $\bar{f}'' = \frac{\alpha}{c} + (1+\epsilon)V_e^{+\Delta \to 0}(c)$. As a result, $\bar{f}'' > \bar{f}' > (1 + (1-\epsilon)\frac{1}{r-\mu}\mu)\frac{\alpha}{c}$, and we have $Q_2(\bar{f}, c)$ converges to $\frac{\alpha}{c} - \bar{f} + V_e^{+\Delta \to 0}(c)$ uniformly in $c \geq r$ and $\bar{f} \in [\bar{f}', \bar{f}'']$. This implies that $\lim_{\Delta \to 0} Q_2(\bar{f}', c) = \epsilon V_e^{+\Delta \to 0}(c) > 0$ and $\lim_{\Delta \to 0} Q_2(\bar{f}'', c) = -\epsilon V_e^{+\Delta \to 0}(c) < 0$. As a result, there exists some $\bar{\Delta}'' > 0$ such that, for any $\Delta < \bar{\Delta}''$, we can construct a continuous function

$$\bar{f}^{Q_2}(c) \in (\bar{f}', \bar{f}'') = \left(\frac{\alpha}{c} + (1-\epsilon)V_e^{+\Delta \to 0}(c), \frac{\alpha}{c} + (1+\epsilon)V_e^{+\Delta \to 0}(c)\right).$$

In order to establish that there exist $(\bar{f}, c)$ such that two supporting conditions hold, we just need to show that there exists c such that $\bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$. We are going to show that there exists some $\bar{\Delta}$ such that we can find the qualifying c for any $\Delta < \bar{\Delta}$. To show that, we first establish that $\bar{f}^{Q_1}(c) > \bar{f}^{Q_2}(c)$ for c close to $\bar{c}$, and $\bar{f}^{Q_1}(c) < \bar{f}^{Q_2}(c)$ for much larger c. Then we conclude that these two functions must intersect.

To be more specific, given any $c' > r$ and $\bar{f} = \frac{\alpha}{c'} + \frac{1}{2}V_e^{+\Delta \to 0}(c')$, we have

$$\lim_{\Delta \to 0} \frac{Q_1(\bar{f}, c') - 1}{\Delta} = \lim_{\Delta \to 0} \frac{\exp(-r\Delta)(c'\Delta + 1) - 1}{\Delta}\left(1 - \Phi\left(Z\left(\frac{\alpha}{c'}\right)\right)\right) - \lim_{\Delta \to 0} \frac{\Phi\left(Z\left(\frac{\alpha}{c'}\right)\right)}{\Delta}$$
$$= c' - r > 0.$$

Therefore, for any $c' > r$, we can have $\bar{\Delta}(c')$ such that for any $\Delta < \bar{\Delta}(c')$, Assumption 1 is satisfied and

$$\bar{f}^{Q_1}(c') < \frac{\alpha}{c'} + \frac{1}{2}V_e^{+\Delta \to 0}(c') < \frac{\alpha}{c'} + (1-\epsilon)V_e^{+\Delta \to 0}(c') < \bar{f}^{Q_2}(c').$$

Intuitively, when $\Delta \to 0$, $\bar{f}^{Q_1}(c')$ should be very close to $\frac{\alpha}{c'}$. Otherwise, given that it is very unlikely to have a large negative shock and $c' > r$, the present value of the debt is larger than 1. On the other hand, $\bar{f}^{Q_2}(c')$ converges to $\frac{\alpha}{c'} + V_e^{+\Delta \to 0}(c')$. Therefore, $\bar{f}^{Q_2}(c') > \bar{f}^{Q_1}(c')$.

In addition, $Q_1(\bar{f}, \bar{c}) < 1$ for any $\bar{f}$ and Δ. Therefore, for any $\Delta < \bar{\Delta}(c')$ and $\bar{f} = \frac{\alpha}{r} + \frac{3}{2}V_e^{+\Delta \to 0}(r)$, we can find $c'' \to \bar{c}$ such that $Q_1(\bar{f}, c'') < 1$. As a result, we have

$$\bar{f}^{Q_1}(c'') > \frac{\alpha}{r} + \frac{3}{2}V_e^{+\Delta \to 0}(r) > \frac{\alpha}{c''} + (1+\epsilon)V_e^{+\Delta \to 0}(c'') > \bar{f}^{Q_2}(c'').$$

Intuitively, when $c'' \to \bar{c}$, the firm should be very unlikely to default to keep the present value of debt equal to 1. In this case $\bar{f}^{Q_1}(c'')$ should be way larger than $\bar{f}^{Q_2}(c'')$ which is close to $\frac{\alpha}{c''} + V_e^{+\Delta\to 0}(c'')$.

By the continuity of $\bar{f}^{Q_1}(c)$ and $\bar{f}^{Q_2}(c)$, we can have $c \in (c'', c')$ such that $\bar{f}^{Q_1}(c) = \bar{f}^{Q_2}(c)$. As a result, we can find $\bar{\Delta}$ such that for any $\Delta < \bar{\Delta}$, there exists $(\bar{f}, c)$ such that Equations (17) and (19) hold.

6.4 Proof of Lemma 3

By deviating to $\gamma_t < \gamma(f_t^-) = \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$, the firm alters its payoff through changes in the debt issuance/repurchase component and its continuation value. Other components—such as coupon payments, tax obligations, and maturing debt payments at that period—remain unaffected by the new debt level. As a result, we focus on the components of the payoff that are impacted by the deviation:

$$\begin{aligned}
&\gamma_t P(f_{\gamma_t}^+) + V_e^+\left(f_{\gamma_t}^+\right) = \gamma_t \exp(-r\Delta)\left[1 - \Phi\left(Z(f_{\gamma_t}^+)\right)\right](c\Delta + 1) \\
&+ \exp\left(-r\Delta\right)\left[\Delta\exp(\mu\Delta) - \tau\Delta\exp(\mu\Delta)\left[1 - \Phi\left(Z(f_{\gamma_t}^+) - \sigma\sqrt{\Delta}\right)\right]\right. \\
&+ \tau\Delta\int_{Z(f_{\gamma_t}^+)}^{\infty} \min\left(cf_{\gamma_t}^+, \alpha X_{t+1}\right)\phi(Z_{t+1})dZ_{t+1} - \left(cf_{\gamma_t}^+\Delta + f_{\gamma_t}^+\right)\left[1 - \Phi\left(Z(f_{\gamma_t}^+)\right)\right] \\
&\left.+ \left[\frac{\alpha}{c} + V_e^+\left(\frac{\alpha}{c}\right)\right]\exp(\mu\Delta)\left[1 - \Phi\left(Z(f_{\gamma_t}^+) - \sigma\sqrt{\Delta}\right)\right]\right] \\
&= \exp\left(-r\Delta\right)\left[\Delta\exp(\mu\Delta) - \tau\Delta\exp(\mu\Delta)\left[1 - \Phi\left(Z(f_{\gamma_t}^+) - \sigma\sqrt{\Delta}\right)\right]\right. \\
&+ \tau\Delta\int_{Z(f_{\gamma_t}^+)}^{\infty} \min\left(cf_{\gamma_t}^+, \alpha X_{t+1}\right)\phi(Z_{t+1})dZ_{t+1} - (c\Delta + 1)\exp(-\xi\Delta)f_t^-\left[1 - \Phi\left(Z(f_{\gamma_t}^+)\right)\right] \\
&\left.+ \left[\frac{\alpha}{c} + V_e^+\left(\frac{\alpha}{c}\right)\right]\exp(\mu\Delta)\left[1 - \Phi\left(Z(f_{\gamma_t}^+) - \sigma\sqrt{\Delta}\right)\right]\right]
\end{aligned} \tag{33}$$

Similarly, by adopting the equilibrium strategy $\gamma(f_t^-)$, the firm gets the according value

$$\gamma(f_t^-)P\left(\frac{\alpha}{c}\right) + V_e^+\left(\frac{\alpha}{c}\right).$$

Let us calculate the difference of the firm's value between adopting $\gamma(f_t^-)$ and γ_t and normalize the difference. That is,

$$\frac{\gamma(f_t^-)P\left(\frac{\alpha}{c}\right) + V_e^+\left(\frac{\alpha}{c}\right) - \left(\gamma_t P(f_{\gamma_t}^+) + V_e^+\left(f_{\gamma_t}^+\right)\right)}{\Delta\left(\frac{\alpha}{c} - f_{\gamma_t}^+\right)}$$

It is important to carefully calculate the change in the tax-shield term.

$$
\begin{aligned}
&\tau\Delta \int_{Z(\frac{\alpha}{c})}^{\infty} \min\left(\alpha, \alpha X_{t+1}\right) \phi(Z_{t+1}) dZ_{t+1} - \tau\Delta \int_{Z(f^+_{\gamma_t})}^{\infty} \min\left(c f^+_{\gamma_t}, \alpha X_{t+1}\right) \phi(Z_{t+1}) dZ_{t+1} \\
&= -\tau\Delta \int_{Z(f^+_{\gamma_t})}^{Z(\frac{\alpha}{c})} \min\left(c f^+_{\gamma_t}, \alpha X_{t+1}\right) \phi(Z_{t+1}) dZ_{t+1} \\
&+ \tau\Delta \int_{Z(\frac{\alpha}{c})}^{Z_c(\frac{\alpha}{c})} \left[\alpha X_{t+1} - \min\left(c f^+_{\gamma_t}, \alpha X_{t+1}\right)\right] \phi(Z_{t+1}) dZ_{t+1} + \tau\Delta \int_{Z_c(\frac{\alpha}{c})}^{\infty} \left(\alpha - c f^+_{\gamma_t}\right) \phi(Z_{t+1}) dZ_{t+1} \\
&> -\tau\Delta \int_{Z(f^+_{\gamma_t})}^{Z(\frac{\alpha}{c})} c f^+_{\gamma_t} \phi(Z_{t+1}) dZ_{t+1} + \tau\Delta \int_{Z_c(\frac{\alpha}{c})}^{\infty} \left(\alpha - c f^+_{\gamma_t}\right) \phi(Z_{t+1}) dZ_{t+1} \\
&= -\tau\Delta c f^+_{\gamma_t} \left(N\left(Z\left(\frac{\alpha}{c}\right)\right) - N\left(Z(f^+_{\gamma_t})\right)\right) + \tau\Delta\left(\alpha - c f^+_{\gamma_t}\right)\left(1 - N\left(Z_c\left(\frac{\alpha}{c}\right)\right)\right).
\end{aligned}
$$

After normalization, we have

$$
\lim_{\Delta\to 0} \frac{\tau\Delta c f^+_{\gamma_t}\left(N\left(Z\left(\frac{\alpha}{c}\right)\right) - N\left(Z(f^+_{\gamma_t})\right)\right)}{\Delta\left(\frac{\alpha}{c} - f^+_{\gamma_t}\right)} = 0, \quad \lim_{\Delta\to 0} \frac{\tau\Delta\left(\alpha - c f^+_{\gamma_t}\right)\left(1 - N\left(Z_c\left(\frac{\alpha}{c}\right)\right)\right)}{\Delta\left(\frac{\alpha}{c} - f^+_{\gamma_t}\right)} = \frac{\tau c}{2}
$$

In addition, we have

$$
\begin{aligned}
&\lim_{\Delta\to 0} \frac{-\tau\Delta \exp(\mu\Delta)\left[1 - \Phi\left(Z\left(\frac{\alpha}{c}\right) - \sigma\sqrt{\Delta}\right)\right] + \tau\Delta\exp(\mu\Delta)\left[1 - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right]}{\Delta\left(\frac{\alpha}{c} - f^+_{\gamma_t}\right)} \\
&= \lim_{\Delta\to 0} \tau\exp(\mu\Delta)\frac{\Phi\left(Z\left(\frac{\alpha}{c}\right) - \sigma\sqrt{\Delta}\right) - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)}{\frac{\alpha}{c} - f^+_{\gamma_t}} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\Delta\to 0} \frac{-\left(c\Delta + 1\right)\exp(-\xi\Delta) f^-_t \left[1 - \Phi\left(Z\left(\frac{\alpha}{c}\right)\right)\right] + \left(c\Delta + 1\right)\exp(-\xi\Delta) f^-_t \left[1 - \Phi\left(Z(f^+_{\gamma_t})\right)\right]}{\Delta\left(\frac{\alpha}{c} - f^+_{\gamma_t}\right)} \\
&= \lim_{\Delta\to 0}\left(c + \frac{1}{\Delta}\right)\exp(-\xi\Delta) f^-_t \frac{\Phi\left(Z\left(\frac{\alpha}{c}\right)\right) - \Phi\left(Z(f^+_{\gamma_t})\right)}{\frac{\alpha}{c} - f^+_{\gamma_t}} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\Delta\to 0}\left[\frac{\alpha}{c} + V^+_e\left(\frac{\alpha}{c}\right)\right]\exp(\mu\Delta)\frac{\left(1 - \Phi\left(Z\left(\frac{\alpha}{c}\right) - \sigma\sqrt{\Delta}\right)\right) - \left(1 - \Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right)\right)}{\Delta\left(\frac{\alpha}{c} - f^+_{\gamma_t}\right)} \\
&= \lim_{\Delta\to 0}\left[\frac{\alpha}{c} + V^+_e\left(\frac{\alpha}{c}\right)\right]\exp(\mu\Delta)\frac{\Phi\left(Z(f^+_{\gamma_t}) - \sigma\sqrt{\Delta}\right) - \Phi\left(Z\left(\frac{\alpha}{c}\right) - \sigma\sqrt{\Delta}\right)}{\Delta\left(\frac{\alpha}{c} - f^+_{\gamma_t}\right)} = 0.
\end{aligned}
$$

As a result,

$$\lim_{\Delta\to 0}\frac{\gamma(f_t^-)P\left(\frac{\alpha}{c}\right)+V_e^+\left(\frac{\alpha}{c}\right)-\left(\gamma_t P(f_{\gamma_t}^+)+V_e^+\left(f_{\gamma_t}^+\right)\right)}{\Delta\left(\frac{\alpha}{c}-f_{\gamma_t}^+\right)}>\frac{\tau c}{2}.$$

Therefore, we show that γ_t is dominated by the equilibrium strategy $\gamma(f_t^-)$ when $\Delta\to 0$.

6.5 Proof of Lemma 4

When $f_t^- < \bar{f}$, the firm gets $\left(-\tau+\tau\min(\alpha, cf_t^-)-cf_t^-\right)\Delta+\frac{\alpha}{c}-f_t^-+V_e^+\left(\frac{\alpha}{c}\right)>0$ by following its equilibrium strategy. If the firm deviates, we have the following cases:

1. If the firm deviates by setting $\gamma_t>\frac{\alpha}{c}-\exp(-\xi\Delta)f_t^-$, the debt-holders will exercise the demand clause so that $d_t=\gamma_t+\exp(-\xi\Delta)f_t^- - \frac{\alpha}{c}$.

   (a) If $\gamma_t\geq\left(-\tau+\tau\min(\alpha, cf_t^-)-cf_t^-\right)\Delta+\frac{\alpha}{c}-f_t^-+V_e^+\left(\frac{\alpha}{c}\right)$, the firm subsequently default. Anticipating this, the market prices $\tilde{P}_t'=0$ for γ_t and therefore, the firm gets 0 from this deviation, which means that this deviation is not profitable.

   (b) If $\gamma_t<\left(-\tau+\tau\min(\alpha, cf_t^-)-cf_t^-\right)\Delta+\frac{\alpha}{c}-f_t^-+V_e^+\left(\frac{\alpha}{c}\right)$, the firm will not default. Anticipating this, the market prices $\tilde{P}_t'=1$ for γ_t. As a result, the firm gets

   $$\begin{aligned}&\gamma_t-d_t+\left(-\tau+\tau\min(\alpha, cf_t^-)-cf_t^-\right)\Delta+\left(\exp(-\xi\Delta)-1\right)f_t^-+V_e^+\left(\frac{\alpha}{c}\right)\\&=\left(-\tau+\tau\min(\alpha, cf_t^-)-cf_t^-\right)\Delta+\frac{\alpha}{c}-f_t^-+V_e^+\left(\frac{\alpha}{c}\right)\end{aligned}$$

   from this deviation, which equals its payoff from following the equilibrium strategy.

2. If the firm deviates by setting $\gamma_t<\frac{\alpha}{c}-\exp(-\xi\Delta)f_t^-$, then, as shown in Lemma 3, such a deviation effectively reduces the tax shield benefit without generating sufficient savings on distress costs as $\Delta\to 0$. As a result, this deviation is not profitable.

When $f_t^-\geq\bar{f}$, the firm will default and receive zero under its equilibrium strategy. If the firm deviates, we consider the following cases:

1. If $\gamma_t>\frac{\alpha}{c}-\exp(-\xi\Delta)f_t^-$, the debt-holders will exercise the demand clause, and the payout is $d_t=\gamma_t+\exp(-\xi\Delta)f_t^- - \frac{\alpha}{c}$.

   (a) If $\gamma_t<\left(-\tau+\tau\min(\alpha, cf_t^-)-cf_t^-\right)\Delta+\frac{\alpha}{c}-f_t^-+V_e^+\left(\frac{\alpha}{c}\right)$, the firm will not default under the equilibrium strategy, and the market prices $\tilde{P}_t'=1$ for γ_t. As a result, the firm receives

$$\begin{aligned}&\gamma_t - d_t + \left(-\tau + \tau\min(\alpha, cf_t^-) - cf_t^-\right)\Delta + \left(\exp(-\xi\Delta) - 1\right) f_t^- + V_e^+\left(\frac{\alpha}{c}\right)\\&= \left(-\tau + \tau\min(\alpha, cf_t^-) - cf_t^-\right)\Delta + \frac{\alpha}{c} - f_t^- + V_e^+\left(\frac{\alpha}{c}\right) \le 0,\end{aligned}$$

where the inequality follows from the Supporting Condition for Optimal Default, Equation (19). Thus, the deviation is not profitable.

(b) If $\gamma_t \ge \left(-\tau + \tau\min(\alpha, cf_t^-) - cf_t^-\right)\Delta + \frac{\alpha}{c} - f_t^- + V_e^+\left(\frac{\alpha}{c}\right)$, the firm will default, and the market prices $\tilde{P}_t' = 0$ for γ_t. As a result, the firm receives zero and does not benefit from the deviation.

2. If $\gamma_t \le \frac{\alpha}{c} - \exp(-\xi\Delta)f_t^-$, the debt-holders do not exercise the demand clause, so $d_t = 0$.

(a) If

$$\left(-\tau + \tau\min(\alpha, cf_t^-) - cf_t^-\right)\Delta + \left(1 - \exp(-\xi\Delta)\right) f_t^- + V_e^+\left(\exp(-\xi\Delta)f_t^- + \gamma_t\right) > 0,$$

the firm will not default under the equilibrium strategy, and the market prices $\tilde{P}_t' = 1$ for γ_t. The firm then receives

$$\begin{aligned}&\gamma_t + \left(-\tau + \tau\min(\alpha, cf_t^-) - cf_t^-\right)\Delta + \left(\exp(-\xi\Delta) - 1\right) f_t^- + V_e^+\left(\exp(-\xi\Delta)f_t^- + \gamma_t\right)\\&< \left(-\tau + \tau\min(\alpha, cf_t^-) - cf_t^-\right)\Delta + \frac{\alpha}{c} - f_t^- + V_e^+\left(\frac{\alpha}{c}\right) \le 0,\end{aligned}$$

where the first inequality follows from a similar calculation as in Lemma (3). Thus, the deviation is not profitable.

(b) If

$$\left(-\tau + \tau\min(\alpha, cf_t^-) - cf_t^-\right)\Delta + \left(1 - \exp(-\xi\Delta)\right) f_t^- + V_e^+\left(\exp(-\xi\Delta)f_t^- + \gamma_t\right) \le 0,$$

the firm will default under the equilibrium strategy, and the market prices $\tilde{P}_t' = 0$ for γ_t. As a result, the firm receives zero, and the deviation is not profitable.

6.6 Proof of Lemma 5

By the proof of Lemma 2, we have $\lim_{\Delta\to 0} f(\Delta) = \frac{\alpha}{c(\Delta)} + V_e^{+\Delta\to 0}(c(\Delta))$ because of construction of $\bar{f}^{Q_2}(c)$.

In addition, for any $\bar{c}' > r$, there exists a $\bar{\Delta}$ such that for all $\Delta < \bar{\Delta}$ and $c \ge \bar{c}'$, $Q_1(\frac{\alpha}{c} + V_e^{+\Delta\to 0}(c), c) > 1$. This implies that for all $\Delta < \bar{\Delta}$, $c(\Delta) \in (r, \bar{c}')$. As a result, when $\Delta \to 0$, $c(\Delta) \to r$, and $\bar{f}(\Delta) \to \frac{\alpha}{r} + V_e^{+\Delta\to 0}(r)$.

6.7 Proof of Proposition 2

Given any $c(\Delta) \to r$ and $\bar{f}(\Delta) \to \frac{\alpha}{r} + V_e^{+\Delta\to 0}(r)$, we have

$$\frac{\frac{\alpha}{c(\Delta)}}{\bar{f}(\Delta)} \to \frac{\frac{\alpha}{r}}{\frac{\alpha}{r} + \frac{1}{r-\mu}(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{r}\mu)} < \frac{1}{1+\frac{\mu}{r-\mu}}. \tag{34}$$

This implies that

$$Z\left(\frac{\alpha}{c(\Delta)}\right) = \frac{1}{\sigma\sqrt{\Delta}}\left(\log\frac{\frac{\alpha}{c(\Delta)}}{\bar{f}(\Delta)} - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right) \to -\infty. \tag{35}$$

Importantly, this converges at the same speed as $\frac{1}{\sqrt{\Delta}}$.

As a result, we have the probability of bankruptcy converges to 0 very fast. That is,

$$\lim_{\Delta\to 0}\frac{\Phi\left(Z\left(\frac{\alpha}{c(\Delta)}\right)\right)}{\Delta} = 0.$$

Similarly, we have

$$\lim_{\Delta\to 0}\frac{\Phi\left(Z\left(\frac{\alpha}{c(\Delta)}\right) + A\sqrt{\Delta}\right)}{\Delta} = 0$$

for any A. As before, we can decompose $V_e^+\left(\frac{\alpha}{c}\right)$ into three parts and derive

$$\lim_{\Delta\to 0} V_e^+\left(\frac{\alpha}{c(\Delta)}\right) = \frac{1}{r-\mu}(1-\tau+\tau\alpha-\alpha+\frac{\alpha}{r}\mu) = \frac{1-(1-\alpha)\tau}{r-\mu} - \frac{\alpha}{r}. \tag{36}$$

In addition, the firm issues $\frac{\alpha}{c(\Delta)} \to \frac{\alpha}{r}$ amount of debt at the price of 1 for any unit of the cash flow. As a result, the firm value per unit of cash flow is $\frac{1-(1-\alpha)\tau}{r-\mu}$.

In addition, let us denote the firm's value, given the time period Δ, when the firm can commit to maintaining a debt level and no default as $V^c(\Delta)$. Since the firm is able to get at most $(1-(1-\alpha)\tau)\, X_t\Delta$ at time t, therefore

$$\begin{aligned} V^c(\Delta) &= \mathbb{E}\left\{\sum_{t=1}\frac{1}{(1+r\Delta)^t}\left(1-(1-\alpha)\tau\right)X_t\Delta|X_0\right\} \\ &= \sum_{t=1}\exp(-r\Delta t)\left(1-(1-\alpha)\tau\right)X_0\exp(\mu\Delta t)\Delta \\ &= (1-(1-\alpha)\tau)X_0\Delta\frac{\exp\left((\mu-r)\Delta\right)}{1-\exp\left((\mu-r)\Delta\right)} \end{aligned} \tag{37}$$

Therefore, we have

$$\lim_{\Delta\to 0} V^c(\Delta) = \frac{(1-(1-\alpha)\tau)X_0}{r-\mu}. \tag{38}$$

Therefore, we conclude that the firm asymptotically achieves its full commitment value as the length of the time period approaches zero.