Dynamic contracting with many agents

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Abstract

We take a mechanism design approach to dynamic capital allocation and risk-sharing between an investor (the principal) and asset managers (the agents). Incentive-compatibility implies that managers with better idiosyncratic performance get larger fees, capital, and continuation utilities. This generates an endogenous distribution of utilities across managers, which is a state variable of the optimal control problem of the principal. With a continuum of agents, this gives rise to a Bellman equation in an infinite-dimensional space, which we solve with mean-field techniques. With CRRA utilities, optimal compensation is proportional to assets-under-management and costly exposure to idiosyncratic risk lowers risky investment.

Keywords: Dynamic contracting, incentives, mechanism design, risk sharing, capital allocation, delegated asset management, continuous time, Martingales, mean-field control

1 Introduction

How should capital be allocated among investment projects? And how should the risk of these projects be shared among the agents managing them? In a frictionless world, capital allocation would be driven by risk and return considerations, as in Merton (1969), and risk-sharing would entail mutualization of diversifiable risk, as in Borch (1962). Frictions, however, can lead to deviations from such optimal allocations. This paper considers an important friction: asymmetric information between investors and agents managing their assets. Thus, starting from a dynamic consumption and portfolio choice problem à la Merton (1969), we study how information asymmetry alters the dynamics of optimal capital allocation and risk-sharing.

An important example of the situation analyzed in this paper is the delegation of investments by large investors, e.g., sovereign wealth funds, family offices, or pension funds, to agents investing in productive assets, e.g., private equity fund managers, real estate fund managers, or infrastructure fund managers. For example, at the end of 2023, the Norwegian sovereign wealth fund had 742 billion kroner (approximately 67 billion US dollars), managed externally by 103 organisations.[1] Rose (2016) notes that "nearly all sovereign wealth funds make significant use of external managers, particularly for alternative asset classes that are difficult to access or demand labor intensive investment strategies, such as private equity, venture capital, and hedge fund strategies." This quote suggests that delegation is motivated by the need to rely on the skills of the managers, skills which the investor does not master. In fact, in the Invesco Global Sovereign Asset Management Study, respon-

[1] See https://www.nbim.no/en/the-fund/how-we-invest/external-mandates. The Norwegian Ministry of Finance decides the allocation of the portfolio across broadly defined asset classes, and then the sovereign fund decides how to allocate funds within these classes to different asset managers.

dents stated they often relied on external asset managers, and one of the respondents declared:[2] "When... we have been forced to operate the business ... that has historically not worked well, so we are moving away from that model and trying to leave all the management to general partners". However, when delegation is motivated by superior understanding of assets by agents, information is likely to be asymmetric between the investor and the asset managers.

To model information asymmetry between the investor and the asset managers, one possible specification is that agents privately observe their return and can secretly divert and consume part of it, as in Bolton and Scharfstein (1990). Another possible specification is that agents privately observe the costly effort they exert to improve expected returns, as in Holmström and Tirole (1997). In both cases, the contract must be designed under the incentive constraint that agents "do the right thing", that is, fully reveal privately observed performance or implement the effort required by the principal. For simplicity, in the paper, we focus on the first specification (privately observed performance), but, as explained below, a specification of the privately observed effort model in line with Holmström and Tirole (1997) leads to similar results as the privately observed return model.[3]

We consider a continuous-time, infinite-horizon economy with a risk-averse investor, in line with Merton (1969). Our economy, however, is a real production economy. There is a single good, which can be consumed or invested as capital in N production technologies. Moreover, we assume the principal cannot operate the technologies and must delegate their management to risk-

[2] See https://ioandc.com/sovereign-wealth-funds-tap-external-managers-for-private-markets-payday/

[3] This is in line with DeMarzo and Fishman (2007a) and DeMarzo and Sannikov (2006) who consider a specification in which returns are privately observed by the agent and show it leads to similar results as a specification in which effort is privately observed by the agent.

averse agents. Each agent operates his/her own production technology, which can be interpreted as the technology for which the agent has the expertise which the principal lacks. The output generated by each agent is proportional to the amount of capital under his/her management. For simplicity, all agents are equally skilled, i.e., they have the same expected return per unit. They, however, are subject to different random shocks: each agent's return is hit by an agent-specific idiosyncratic shock, as well as a common aggregate shock. The aggregate shock and the allocation of capital across agents are publicly observable, and capital allocation is decided by the principal. In contrast, each agent privately observes his/her output and can secretly divert and consume some of it.

Applying the revelation principle, we study revelation mechanisms in which incentive compatibility constraints are imposed, so that when agents privately observe their return, they truthfully report it to the principal. To analyze incentive constraints, we rely on the martingale techniques introduced in contract theory by Sannikov (2008). Incentive compatibility implies that an agent's utility must be sensitive to his/her idiosyncratic shock. Intuitively, after good performance agents are rewarded, while after bad performance they are punished. Differences in performance, therefore, generate endogenous heterogeneity among agents.

In order to solve the optimal control problem of the principal, we consider the limit case in which the number of agents goes to infinity. This simplifies the problem, because in this case, each agent's idiosyncratic shock is negligible relative to aggregate output. Not only does this imply that diversification washes away the impact of individual shocks on aggregate output, it also implies that making an agent's reward sensitive to this agent's idiosyncratic shock does not affect the aggregate resources available for the other agents.

This simplification enables us to obtain, for CRRA utilities, an analytical characterization of the optimal mechanism under information asymmetry.

However, considering a continuum of agents also generates some mathematical difficulties. The distribution of the continuation utilities of the different agents is a state variable of the control problem of the principal. With a finite number of agents, the information contained by this state variable is simply the vector of continuation utilities of the different agents. With a continuum of agents, the distribution of utilities across agents is an infinite dimensional object. And, with aggregate shocks, it evolves stochastically. To analyze the optimal mechanism in that context we need to perform Itô calculus with infinite dimensional variables. To do so we rely on Mean Field techniques.[4] This enables us to obtain the following economic insights:

- The incentive compatibility constraint implies that agents with better (resp. worse) idiosyncratic performance are rewarded (resp. punished), although, on the equilibrium path, performance only reflects luck. This generates exposure to idiosyncratic risk, reducing welfare relative to the first-best in which idiosyncratic risk is fully mutualized, as in Borch (1962).

- Although the link between performance and rewards is a common finding in the literature, what is new in the present paper is that this link goes through capital allocation. Agents with good performance are allocated more capital than agents with bad performance.[5] So capital allocation among agents plays a key role in providing incentives.

[4] In physics, Mean Field Theory studies the behavior of models where a large number of particles interact randomly by studying a simpler model where each particle interacts with an average distribution (a mean-field) of particles.

[5] This allocation rule is optimal in spite of the fact that performance just reflects luck, and does not reflect any advantage in terms of skills and productivity.

- Incentives do not only affect how capital is allocated among agents, they also affect the overall amount of capital delegated to agents. In fact, they lower overall investment. The intuition is the following: Because of incentive constraints, risk from capital investment is imperfectly shared among agents. Correspondingly, holding capital comes at the cost of greater risk exposure than in the first best. The greater cost of holding capital implies that it is optimal to undertake less capital investment than in the first best.

- In the optimal mechanism, capital allocation becomes more and more heterogeneous with time, with successful managers being allocated a larger fraction of capital than unsuccessful ones. Thus incentives breed inequality.

- The sensitivity of agents to their idiosyncratic risk in the optimal mechanism depends on the agents' bargaining power. When agents have little bargaining power, the principal can design the contract to extract a lot of rents from them. To do so, the principal makes the agents' utilities very sensitive to their idiosyncratic risk, which relaxes the incentive constraint. So, as the bargaining power of the agents declines, the volatility of their compensation increases.

Section 2 discusses the relation between our paper and the literature. Section 3 presents the model. Section 4 considers the symmetric information benchmark, characterizing the first-best optimal allocations. Section 5 turns to the case of asymmetric information, characterizing second-best optimal allocations. Section 6 presents the explicit solution obtained in the logarithmic utility case. Section 7 concludes. Proofs not provided in the text are in the appendix.

2 Literature

Our analysis is at the meet of three strands of the continuous-time literature in finance:

- First, we build on the dynamic optimal contracting literature studying corporate finance and financial intermediation, in line with the seminal contributions of DeMarzo and Fishman (2007a, 2007b) and Sannikov (2008).[6] A key difference between our paper and that literature is that we study the optimal contract between the principal and *several* agents. Thus, the issue of how to allocate capital and share risk among agents lies at the heart of our analysis.

- Second, we also build on the application of Bellman's principle to portfolio choice by Merton (1969, 1973, 1987). The contribution of our analysis relative to that literature is to allow for asymmetric information between investors and asset managers, implying that capital allocation is not only driven by risk and return considerations, but also by incentive considerations.

- Third, our analysis is related to the mean-field literature, see, e.g., Lasry and Lions (2007), Cardaliaguet (2012), and Carmona and Delarue (2018). More precisely, we study a Mean Field Control problem.

[6]See also DeMarzo and Sannikov(2006), Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Sannikov (2006), Biais, Mariotti, Rochet, Villeneuve (2010), DeMarzo, Fishman, He and Wang (2012), Feng and Westerfield (2021), Yang (2020), Di Tella and Sannikov (2021), Dai, Wang, Yang (2021), and Gryglewicz and Mayer (2023). In the present paper, as in Biais, Mariotti, Rochet, and Villeneuve (2010) and DeMarzo, Fishman, He, and Wang (2012), the scale of operations is determined by the optimal contract and is useful to provide incentives. However, in contrast with Biais, Mariotti, Rochet, Villeneuve (2010) and DeMarzo, Fishman, He and Wang (2012), in the present paper there are no capital adjustment costs. This enhances tractability and gives rise to continuous reallocation of capital. He (2009) offers an interesting alternative approach in which firm size is affected by unobservable agent's effort. This differs from our model in which firm size is directly controlled by the principal, and what is unobservable is output.

> This is related to, but different from, Mean Field Games. Mean Field Games have been introduced in macroeconomics by Achdou, Buera, Lasry, Lions, and Moll (2022). In their model, each agent solves an optimal control problem associated with a Bellman equation, while the distribution of wealth among agents is determined by a Fokker-Planck equation. In our analysis, there is a single control problem, that of the planner who controls the distribution of individual continuation utilities. That control problem, whose state variables involve the infinite-dimensional distribution of agents' utilities, is a mean-field control problem.

Our paper is related to the literature on delegated asset management. Our theoretical finding that asset managers with better performance attract more capital, and that asset management fees are proportional to assets under management, are in line with stylized facts and empirical evidence (see Chevalier and Ellison (1997) and Sirri and Tufano (1998)). Together with our theoretical finding that the distribution of assets across managers is skewed to the right, this implies that the distribution of fees is skewed to the right, in line with empirical findings (see Bai et al, 2024). Our work is also related to the insightful analyses of optimal contracting between an investor and an asset manager by He and Xiong (2013) and Ou-Yang (2003). Binsbergen, Brandt and Koijen (2008) and Berk and Green (2004) also study the interaction between investors and asset managers, but do not analyze the optimal contract between the former and the latter. Our paper differs from Binsbergen, Brandt and Koijen (2008), He and Xiong (2013), and Ou-Yang (2003) by examining the case in which an investor hires several asset managers and dynamically reallocates capital among agents. In He and Xiong (2013) and Ou-Yang (2003) there is only one agent, and capital is allocated once and for all at the beginning of the contract. In Binsbergen, Brandt and Koijen (2008) there are several agents but, when the opportunity set is constant (as in our paper), the investor chooses a fixed allocation of capital across managers (unlike in our paper). Thus, a contribution of our paper is to study how incentive constraints affect dynamic capital allocation. In Berk and Green (2004) also, capital is allocated dynamically: As they observe managers' returns, symmetrically informed principals and agents update their beliefs about managers' skills. Because it signals high skills, good performance attracts investors, but, due to decreasing returns to scale, the increase in assets under management reduces profitability, so that returns are not persistent. In our model also, past performance attracts investment flows, while returns are not persistent, but this is for different reasons: In our constant returns-to-scale model, unlike in Berk and Green (2004), it is common knowledge that all agents are equally skilled and good performance is only due to luck. In our analysis, allocating more capital to an agent with good performance is a way to reward this agent for truthfully revealing good performance, which is required by incentive compatibility. Thus, our approach differs from that of Berk and Green (2004), who write: "If all performance is due to luck, there should be no reason to reward it."

As in Holmström (1982), the principal in our paper interacts with several privately informed agents. However, in Holmström (1982), the actions of the agents interact in the determination of aggregate output. In contrast in our paper, agents' individual returns are independent from one another (conditional on the aggregate shock), so the interaction among agents only goes via resource constraints, in particular the constraint regarding the allocation of capital at a given point in time. As mentioned above, for incentive reasons, the capital allocated to each agent must reflect this agent's idiosyncratic shock. So, with a finite number of agents, the idiosyncratic shock of one agent, by affecting how much capital is allocated to that agent, could affect other agents, via the resource constraint. However, with a continuum of agents, agents are negligible relative to aggregate quantities. So, the relation between one agent's shock and this agent's capital allocation does not affect the other agents. This simplifies the problem and helps us obtain, for CRRA utility, an explicit solution for the optimal mechanism.

Our paper is also related to the macro-theory literature on dynamic heterogeneity in constrained economies, see, e.g., Aiyagari (1994), Angeletos (2007), Achdou et al (2022), Bewley (1977), Huguett (1997), and Krusell and Smith (1998). These models and ours share the property that idiosyncratic risks cannot be fully insured, implying that the dynamics of capital and consumption depend on the whole distribution of wealth across agents, an infinite-dimensional variable. This generates technical difficulties, which Krusell and Smith (1998) have approached by *assuming* that agents' expectations only depend on a one-dimensional statistics, the average of the wealth distribution. They calibrate this "approximated equilibrium" and show that it fits the US consumption and investment data. In contrast, we *prove* that in our model, when utilities are CRRA, constrained optimal allocations only depend on one-dimensional statistics.

Aiyagari (1994), Angeletos (2007), Achdou et al (2022), Bewley (1977), Huguett (1997), and Krusell and Smith (1998) consider exogenous financial constraints: The only available financial instrument is a risk-free bond, and agents' wealth cannot go below a given threshold. This constraint binds only at the boundary, which generates a Dirac mass in the distribution of wealth. In contrast, our paper provides a microfoundation of the impossibility of fully insuring idiosyncratic risks, namely, the incentive compatibility constraint.

We show that this constraint binds everywhere, so that the distributions are smooth. Moreover, while these papers look for the stationary distributions, we show that the optimal mechanism does not lead to a stationary distribution. This is similar to Atkeson and Lucas (1992), who also characterize an optimal mechanism. Atkeson and Lucas (1992), however, consider a constant endowment setting, while we consider an investment setting, in which the total amount of resources grows endogenously. Correspondingly, while in Atkeson and Lucas (1992) an agent's utility goes to minus infinity with probability one, this is not the case in our model when productivity is large enough. Another difference is that in Atkeson and Lucas (1992) there is only idiosyncratic risk, while we also consider aggregate risk, which has been a challenge in the literature. The treatment of aggregate risk is an important contribution of our article. It necessitates employing Ito calculus in infinite-dimensional spaces, a technique developed only recently by mathematicians and explained in Carmona and Delarue (2018). Using this technique, we can characterize how aggregate risks are shared between the principal and agents.

3 Model

We extend the portfolio selection model of Merton (1969) to a principal-agent context: the investor (the principal) delegates the management of the portfolio of assets to several managers (the agents).

3.1 Preferences and Technology

Time is continuous: $t \in (0, \infty)$. The agents, indexed by $i = 1, ...N$, and the principal are infinitely lived with discount rate ρ and utility from consumption c equal to $\rho u(c)$, where $u(.)$ is increasing and concave.[7] To obtain closed form

[7]Multiplying current utility by ρ is a useful convention that generates simpler formulas. For example, the intertemporal utility of a constant consumption flow c is

solutions, we will subsequently assume that u is CRRA: $u(c) = \log c$ or $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$ with $\alpha > 0$ and different from 1. However, our general characterization results are valid for general utility functions.

Each of the N agents has mass $1/N$, so that the total mass of the agents is equal to 1. We will later take the limit when N goes to infinity to consider a continuum of agents. There is a single good which can be consumed or invested in N stochastic constant returns to scale technologies.[8] The principal does not have the expertise to operate the production technologies, but the agents do. More precisely, each agent has the expertise to operate one of the N technologies. If agent i invests an amount $\frac{k_t^i}{N}$ in his/her own technology, this generates instantaneous output:[9]

$$dY_t^i = \frac{k_t^i}{N}[\mu dt + \sigma dZ_t^i + \sigma_A dZ_t^A], \tag{1}$$

where μ is the expected rate of return of the technology and $(Z_t^A, Z_t^i)_{i=1,...N}$, are independent Brownian motions. (dZ_t^A) is an aggregate productivity shock, to which all projects are equally exposed, while $(dZ_t^i)_{i=1,...N}$ are idiosyncratic, project-specific, productivity shocks. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the augmented filtration generated by the N+1-dimensional Brownian motion $(Z_t^A, Z_t^1, \ldots, Z_t^N)_{t\geq 0}$. All processes introduced in this section are assumed to be square-integrable and progressively measurable with respect to $(\mathcal{F}_t)$.

just $\int_0^\infty \rho e^{-\rho t} u(c)dt = u(c)$.

[8]With constant returns to scale, principal's risk-aversion is important for interior solutions. If the principal had linear utility, the problem would be degenerate, with either 0 investment or infinite investment.

[9]We denote by $\frac{k_t^i}{N}$ rather than k_t^i the capital invested by agent i. This makes easier the comparison with the case $N = \infty$.

3.2 Capital

The total amount of capital in the economy at time t is

$$K_t := \frac{1}{N}\sum_{i=1}^{N} k_t^i. \tag{2}$$

If, at time t, the consumption flow of agent i is $\frac{c_t^i}{N}$ and that of the principal is c_t^P, the law of motion of aggregate capital is

$$dK_t = \sum_{i=1}^{N} dY_t^i - \left(\frac{1}{N}\sum_{i=1}^{N} c_t^i + c_t^P\right) dt.$$

Substituting (1) this is

$$dK_t = \left(\mu K_t - \frac{1}{N}\sum_{i=1}^{N} c_t^i - c_t^P\right) dt + \sigma_A K_t dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^{N} k_t^i dZ_t^i. \tag{3}$$

Equation (3) is a resource constraint stating that net aggregate investment is equal to total output (net of depreciation) minus total consumption. Note that, since the idiosyncratic shocks are independent, when N goes to infinity the last term in (3) vanishes[10]. That is, idiosyncratic shocks cancel, and only the aggregate productivity shock matters for aggregate output.

3.3 Pareto frontier

A consumption path is a progressive measurable nonnegative process $(c_t)_t$ such that

$$\mathbb{E}\int_0^{\infty} \rho e^{-\rho s}|u(c_s)|ds < +\infty.$$

[10]The variance of this term per unit of time is $\frac{\sigma^2}{N^2}\sum_{i=1}^{N}(k_t^i)^2$. This converges to zero when the allocation of individual capital is square integrable.

For a given consumption path (c_t^i), agent i's expected continuation utility at time t is

$$\omega_t^i := \mathbb{E}_t \int_t^\infty \rho e^{-\rho(s-t)} u(c_s^i) ds. \tag{4}$$

We seek to characterize the Pareto frontier of the economy by computing the value function V of the principal, defined as the maximum expected utility she can obtain with a total amount of capital K when agent $i \in (1, ..., N)$ gets initial expected utility ω^i. Let us denote by $\mathbb{W}$, the vector of initial expected utilities $(\omega^1, ..., \omega^N)$. The value function of the principal is obtained by finding capital and consumption paths (k_t^i, c_t^i, c_t^P) that maximize the principal's expected utility

$$\mathbb{E} \int_0^\infty \rho e^{-\rho t} u(c_t^P) dt, \tag{5}$$

subject to the relevant constraints. In symmetric information, these constraints are the capital allocation constraint (2), the resource constraint (3), and the initial conditions: $K_0 = K$, and $\omega_0^i = \omega^i$, for $i = 1, ..., N$. Under information asymmetry, there will also be an incentive compatibility constraint.

We assume that agents and principal commit to the contract signed at date 0.[11] We first consider the case of symmetric information, in which idiosyncratic shocks and thus individual outputs are publicly observable. This case will then serve as a benchmark for the case in which agents privately observe their idiosyncratic shocks.

3.4 Promise keeping

Denote by M_t^i the expectation of the lifetime utility of agent i conditional on time t information:

$$M_t^i \equiv \mathbb{E}_t \left[\int_0^\infty \rho e^{-\rho s} u(c_s^i) ds \right] = \int_0^t \rho e^{-\rho s} u(c_s^i) ds + e^{-\rho t} \omega_t^i.$$

[11] Hence, we do not specify outside options for the agents beyond time 0.

Consequently, the dynamics of ω_t^i is

$$d\omega_t^i = \rho \left[\omega_t^i - u(c_t^i)\right] dt + e^{\rho t} dM_t^i.$$

Since M_t^i is the conditional expectation of an integrable random variable, it is a martingale. By the Martingale Representation Theorem, the martingale term in $d\omega_t^i$ can be written as a linear combination of the Brownian shocks at time t, $(dZ_t^i)_{i=1,...,N}$ and (dZ_t^A). Thus the dynamics of i's continuation utility can be written as

$$d\omega_t^i = \rho \left[\omega_t^i - u(c_t^i)\right] dt + \frac{\sigma}{N} \sum_j \beta_t^{ij} dZ_t^j + \sigma_A \beta_t^{A,i} dZ_t^A, \tag{6}$$

where β_t^{ij} and $\beta_t^{A,i}$ are adapted to $(\mathcal{F}_t)$. $\beta_t^{A,i}$ can be interpreted as the share of the volatility of the aggregate shock that is borne by agent i, while β_t^{ij} is the exposure of agent i to the idiosyncratic shock of agent j. In line with the contract theory literature, hereafter we refer to equation (6) as the "promise keeping" condition.

4 Optimal allocations under symmetric information

The focus of this paper is on the asymmetric information case. However, the symmetric information case offers a useful benchmark in which our methodological approach can be explained in simple terms, paving the way to the more complex asymmetric information case.

4.1 The control problem of the principal

The state variables of the principal's control problem are the capital stock K_t and the vector $\mathbb{W}_t \equiv (\omega_t^1, ..., \omega_t^N)$ of promised utilities. The controls k_t^i, c_t^i, c_t^P,

β_t^{ij}, and $\beta_t^{A,i}$ only depend on K_t and $\mathbb{W}_t$. The value function of the principal is

$$V(K,\mathbb{W}) = \sup_{k,\beta,\beta^A,c,c^P} \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c_t^P)dt\right], \tag{7}$$

subject to the capital allocation constraint (2), the resource constraint (3), the promise keeping condition (6), and the initial conditions: $K_0 = K$ and $\mathbb{W}_0 = \mathbb{W}$. The associated Hamilton-Jacobi-Bellman equation is

$$\rho V(K,\mathbb{W}) = \sup_{k,\beta,\beta^A,c,c^P} \rho u(c^P) + \sum_{i=1}^N \left[V_{\omega^i}\rho\left(\omega^i - u(c^i)\right)\right] + V_K\left(\mu K - c^P - \sum_{i=1}^N \frac{c^i}{N}\right)$$

$$+\sum_{i=1}^N \left[V_{\omega^i K}\left(\beta^{A,i}\sigma_A^2 K + \sigma^2 \sum_j \frac{k^j\beta^{ij}}{N^2}\right)\right] + \frac{1}{2}V_{KK}\left[\sigma_A^2K^2 + \sigma^2\sum_i\left(\frac{k^i}{N}\right)^2\right]$$

$$+\frac{1}{2}\sum_{i,j} V_{\omega_i\omega_j}\left(\sigma_A^2\beta^{A,i}\beta^{A,j} + \sigma^2\sum_\ell \frac{\beta^{i\ell}\beta^{j\ell}}{N^2}\right) \tag{8}$$

subject to the capital allocation constraint (2).

The shape of the Hamilton-Jacobi-Bellman equation (8) implies that in the first-best, consumption allocation and capital allocation are separable. The consumption allocation controls c^i and c^P appear only in the first line of (8), while the capital allocation controls k^i only appear in the second line of (8).

The standard verification theorem in stochastic control (see, e.g., Yong and Zhou (1999) chapter 4) yields our first proposition:

Proposition 1 *If there exists a twice continuously differentiable solution V of the HJB equation (8) that satisfies the transversality condition*

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}[V(K_t,\mathbb{W}_t)] = 0,$$

it is the value function of the principal's problem (7).

We now show that, when u is CRRA, the solution is in closed form.

4.2 An explicit solution when utility is CRRA

Merton (1969) studied a simpler form of our model, in which the investor does not need agents and directly manages the portfolio of assets. When u is CRRA, and assets have i.i.d. returns with expectation μ and volatility Σ, the solution exists when

$$\mu < \frac{\rho}{1-\alpha} + \frac{\alpha}{2}\Sigma^2. \tag{9}$$

If condition (9) is not satisfied, the value function is infinite, because the expected return is very large relative to the cost of risk bearing. In contrast, if condition (9) is satisfied, there exists an optimal allocation and it is such that:

- capital is equally allocated across assets,
- the consumption flow is proportional to the capital stock,
- the capital stock follows a geometric Brownian motion,
- the value function of the principal satisfies $V(K) = u(mK)$ for some constant m.

We will now show that a similar result obtains when the principal must delegate asset management to N agents but there is no information asymmetry. To state our result it is useful to denote by A_t the average across agents of the agents' inverse utilities, i.e.,

$$A_t := \frac{1}{N} \sum_i u^{-1}(\omega_t^i).$$

It is useful also to define the constant m such that, when the constant absolute risk aversion index of the agents and the principal is $\alpha \neq 1$,

$$m^{\frac{\alpha-1}{\alpha}} = \frac{1}{\alpha}\rho^{-1/\alpha}(\rho - (1-\alpha)\mu + (1-\alpha)\frac{\alpha}{2}(\sigma_A^2 + \frac{\sigma^2}{N})),$$

while, when the agents and the principal have logarithmic utility

$$m = \rho \exp(\frac{1}{\rho}(\mu - \rho - \frac{1}{2}(\sigma_A^2 + \frac{\sigma^2}{N})).$$

As stated in the following proposition, the first-best allocations between the principal and the agents exhibit similar properties. Relying on these notations, we can present, in our next proposition, the optimal mechanism obtaining under symmetric information:

Proposition 2 *When u is CRRA, (9) holds, and*

$$m > \frac{A_0}{K_0}, \tag{10}$$

then there exists an optimal mechanism. It has the following features:

- *The inverse utility of each agent is proportional to aggregate capital:*

$$u^{-1}(\omega_t^i) = a^i K_t, \tag{11}$$

  where the constant a^i is defined as $\frac{u^{-1}(\omega_0^i)}{K_0}$. Consequently, the average inverse utility of the agents, A_t, is also proportional to aggregate capital.

- *Capital is equally allocated across agents*

$$k_t^i \equiv K_t, i = 1, ..., N, \tag{12}$$

and follows a geometric Brownian motion:

$$\frac{dK_t}{K_t} = \frac{dk_t^i}{k_t^i} = \left(\mu - \gamma^P - \frac{1}{N}\sum_{i=1}^{N}\gamma^i\right)dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^{N} dZ_t^i. \quad (13)$$

- *The consumption flows of the principal and the agents are proportional to aggregate capital stock:*

$$c_t^P = \gamma K_t, c_t^i = \gamma^i K_t, i = 1, ..., N. \quad (14)$$

- *The value function of the principal is*

$$V(K_t, \mathbb{W}_t) = u\left(K_t\left[m - \frac{A_t}{K_t}\right]\right). \quad (15)$$

Condition (10) states that the amount of initial capital is sufficiently large relative to the agents' reservation utilities that it is possible to design a mechanism offering the agents their reservation utility while leaving positive consumption for the principal. Formula (15) shows that the value function depends only on the vector of continuation utilities $\mathbb{W}_t$ through a one-dimensional statistic: A_t. Equation (11) implies that the ratio of A_t to aggregate capital is constant. Similar properties will obtain in the asymmetric information case, which will greatly enhance the tractability of the problem. Proposition 2 illustrates that with CRRA utility and symmetric information, the capital allocation problem and the consumption allocation problem can be solved separately.

- The sole purpose of capital allocation is to maximize diversification. Thus, as stated in Equation (12), capital is allocated equally across projects run by the different agents, to maximize diversification.

- As can be seen in (12) and (14), an agent's consumption and capital allocation only depend on aggregate capital and output, reflecting mutualization of risks. Hence, the individual performance of an agent affects him/her only via its impact on aggregate output. As the number of agents goes to infinity, this impact goes to 0. That the individual performance of an agent equally affects this agent and the other agents is at odds with stylized facts. As shown in the next section, with asymmetric information we will obtain the more plausible result that an agent's performance affects this agent much more than it affects the others.

The proof of Proposition 2 is in the appendix. We adopt the guess-and-verify method that will be used throughout the paper. First, we compute the value function associated with the optimal controls guessed, given in (12), (13), and (14), and show that this yields the value function guessed, given in (15). Second, we verify that this value function satisfies the Bellman equation (8).

Under symmetric information, the inverse utility of each agent is proportional to aggregate capital, as stated in equation (12). In contrast, as will be shown below, with information asymmetry, incentive compatibility constraints prevent inverse utilities from being proportional across agents, since they have to reflect individual performance. However, the proportionality between capital and inverse utilities will be maintained at the **individual** level. Capital allocation will therefore be an important tool for providing incentives to the agents.

5 Optimal allocations under asymmetric information

We now turn to the case in which agents privately observe their idiosyncratic shock (dZ_t^i), while aggregate shocks and individual capital are still publicly observable.[12] Private observation of individual output generates incentive problems because we assume agents can secretly divert and consume some of the output they generate (as in Bolton and Scharfstein (1990) and DeMarzo and Sannikov (2006).)

By the revelation principle, it is without loss of generality to focus on truthful revelation mechanisms. Thus, the second best allocation is achieved by the mechanism optimally mapping the history of aggregate and idiosyncratic shocks into allocations, under the resource constraint and the incentive compatibility constraint that agents truthfully report their shocks.

Heuristically, the sequence of events at time t is the following:

- Each agent i operates the technology, with capital k_t^i.
- Shocks dZ_t^i and dZ_t^A realize. Each agent i reports $d\hat{Z}_t^i$ to the planner, transferring the corresponding output $d\hat{Y}_t^i$.
- As a function of the history of shocks, the planner allocates to the agent i, his/her period consumption flow c_t^i, capital for the next period k_{t+dt}^i, and promised continuation utility ω_{t+dt}^i.

[12] While our analysis assumes agents privately observe their output after it is realized and then can secretly divert it, similar results would obtain in a moral hazard setting à la Holmström and Tirole (1997). In that specification, agents would privately observe if they exert costly effort at time t, and effort would improve the instantaneous drift of their output process. The incentive compatibility condition that the agent exerts the right level of effort leads to a similar incentive compatibility condition to that we obtain. DeMarzo and Sannikov (2006) explicitly demonstrate the similarity between the two forms of information asymmetry in the context of a one-agent model with risk neutrality.

5.1 Incentive compatibility

By the promise keeping condition (6), when truthfully revealing dZ_t^i and anticipating that other agents report truthfully, agent i gets

$$\rho u(c_t^i)dt + \frac{\sigma}{N}\sum_{j=1}^{N}\beta_t^{ij}dZ_t^j + \sigma_A\beta_t^A dZ_t^A. \tag{16}$$

Similarly, when under-reporting: $d\hat{Z}_t^i = dZ_t^i - \delta dt$, agent i gets

$$\rho u(c_t^i + \sigma\delta\frac{k_t^i}{N})dt + \frac{\sigma}{N}\beta_t^{ii}(dZ_t^i - \delta dt) + \frac{\sigma}{N}\sum_{j\neq i}\beta_t^{ij}dZ_t^j + \sigma_A\beta_t^A dZ_t^A. \tag{17}$$

Incentive compatibility requires the former to be larger than the latter. Because u is concave, (16) is larger than (17) if and only if

$$\beta_t^{ii} \geq \rho k_t^i u'(c_t^i). \tag{18}$$

Since $k_t^i u'(c_t^i) > 0$, (18) implies that the dynamics of ω_t^i in (6) must be affected by agent i's idiosyncratic shock.

As shown in the previous section, for logarithmic utility, in the first best, the sensitivity of an agent's continuation utility to his/her idiosyncratic shock was $\beta_t^{ii} = 1$. The first best allocation, however, is not incentive compatible. To see this, substitute c_t^i, k_t^i, and β_t^{ii} from (14), (12), and (54), into (18). This leads to the conclusion that the incentive compatibility holds for the first best allocation with logarithmic utility only if $\gamma^i > \rho$. This, however, is inconsistent with (56). That the first best allocation is not incentive compatible reflects that under asymmetric information agent's consumption, capital, and utility must be more sensitive to the agent's individual performance than in the first best.

The incentive compatibility condition (18) gives rise to a trade-off between risk sharing and investment. Exposing risk-averse agents to idiosyncratic shocks reduces their utility. To reduce that exposure, one needs to reduce β_t^{ii}. Because of (18), however, reducing β_t^{ii} requires increasing c_t^i and/or reducing k_t^i. As shown below, the second best will thus differ from the first best by tilting the consumption trade-off towards consumption, leading to less investment than in the first best.

5.2 The control problem of the principal with a finite number of agents

With asymmetric information, the Hamilton Jacobi Bellman equation of the principal is still given by (8), but now the maximization problem is subject to the incentive compatibility condition (18), in addition to the capital allocation constraint (2).

Condition (18) implies that, in contrast to the first best, capital allocation and consumption allocation decisions are no longer separable. In the HJB equation obtained under symmetric information (8), the terms involving capital k^i and consumption c^i were added to one another. In contrast, in the incentive compatibility condition (18), capital k^i and consumption c^i multiply one another. This makes the optimal control problem much less tractable, and the guess and verify approach difficult.

However, when N goes to infinity, tractability is restored, because each agent's idiosyncratic shock becomes negligible relative to aggregate quantities. So, providing incentives to a given agent, by exposing this agent to his/her own idiosyncratic shock, does not impact the amount of resources left to allocate among the other participants. Regarding the aggregate amount of resources to be allocated, full diversification of idiosyncratic risks is obtained, for arbitrary square integrable allocations of capital. In this context, capital allocation can be used to provide incentives, in complement with consumption allocation. The remainder of the paper is dedicated to the characterization of second best allocations in the case of a continuum of agents.

5.3 The case of a continuum of agents

While restoring tractability, considering a continuum of agents makes the problem more technical because one of the state variables (the distribution of utilities) is infinite dimensional. Moreover, with aggregate risk, this is a stochastic control problem in the space of random measures over $\mathbb{R}$. Therefore we need to use Itô calculus in infinite-dimensional spaces.

With a finite number of agents, the value function is a symmetric function of the vector of utilities $\mathbb{W}$. It can be equivalently expressed as a function of the empirical distribution of agents' utilities. We assume that this empirical distribution weakly converges to a distribution $\mathbb{P}$ when N goes to infinity. Using results from the propagation of chaos theory (Snitzman (1991)), the aggregate consumption of the agents converges to

$$\int c(K_t, \mathbb{P}_t; \omega) d\mathbb{P}_t(\omega), \tag{19}$$

where $\mathbb{P}_t$ denotes the conditional distribution of the agents' utility given the filtration generated by aggregate shocks. In this context, the resource constraint (3) converges to

$$dK_t = \left(\mu K_t - \int c(K_t, \mathbb{P}_t; \omega) d\mathbb{P}_t - c^P(K_t, \mathbb{P}_t)\right) dt + \sigma_A K_t dZ_t^A, \tag{20}$$

which reflects that, when the number of agents goes to infinity, idiosyncratic noise diversifies away, so that the only stochastic term in the dynamics of aggregate capital is the aggregate shock dZ_t^A. Similarly, the dynamics of an agent's continuation utility only contains two stochastic terms, the aggregate shock dZ_t^A and the agent's idiosyncratic shock dZ_t:

$$d\omega_t = \rho\left(\omega_t - u(c_t)\right)dt + \sigma\beta dZ_t + \sigma_A\beta_A dZ_t^A. \tag{21}$$

The mechanism maps i) aggregate capital, ii) the distribution $\mathbb{P}$, and iii) the continuation utility of the agent ω, into the agent's capital allocation, consumption, and sensitivity to shocks:

$$k(K,\mathbb{P},\omega), c(K,\mathbb{P},\omega), \beta_A(K,\mathbb{P},\omega), \beta(K,\mathbb{P},\omega).$$

The control problem of the principal thus writes

$$V(K,\mathbb{P}) = sup_{k,c,c^P,\beta,\beta_A}\mathbb{E}\left[\int_0^\infty \rho e^{-\rho t}u(c_t^P)dt\right], \tag{22}$$

where the expectation is taken over productivity shocks. The state equations are given by the resource constraint and the promise keeping condition, and the maximization is taken over the set of controls that satisfy fo all t the incentive compatibility condition (18) and the capital allocation constraint

$$K_t = \int_{\mathbb{R}} k(K,\mathbb{P};\omega)\, d\mathbb{P}_t(\omega). \tag{23}$$

With a continuum of agents, the principal value function V depends on aggregate capital K and the distribution of continuation utilities $\mathbb{P}$. To write the Hamilton-Jacobi-Bellman equation, we need to apply Itô's formula to V, which, with a continuum of agents, requires differential calculus on the space $\mathcal{P}_2$ of (square integrable) measures on $\mathbb{R}$. To present this analysis, we start by introducing the notions of first and second variations for functions of measures, as in Carmona and Delarue (2018), chapter 4.

Definition 1: *V admits a first variation at $\mathbb{P} \in \mathcal{P}_2(\mathbb{R})$ if there exists a real-valued function*

$$\nabla V : \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$$

such that, for all $\mathbb{Q}$ in $\mathcal{P}_2(\mathbb{R})$, we have

$$\lim_{\varepsilon \to 0} \frac{V(K, (1-\varepsilon)\mathbb{P} + \varepsilon\mathbb{Q}) - V(K, \mathbb{P})}{\varepsilon} = \int_{\mathbb{R}} \nabla V(K, \mathbb{P}, \omega)(d\mathbb{Q} - d\mathbb{P})(\omega). \quad (24)$$

Definition 2: *V admits a second variation at $\mathbb{P} \in \mathcal{P}_2(\mathbb{R})$ if there exists a real-valued function*

$$\nabla^2 V : \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}$$

such that, for all $\mathbb{Q}$ in $\mathcal{P}_2(\mathbb{R})$ and all ω in $\mathbb{R}$, we have

$$\lim_{\varepsilon \to 0} \frac{\nabla V(K, (1-\varepsilon)\mathbb{P} + \varepsilon\mathbb{Q}, \omega) - \nabla V(K, \mathbb{P}, \omega)}{\varepsilon} = \int_{\mathbb{R}} \nabla^2 V(K, \mathbb{P}, \omega, \omega')(d\mathbb{Q} - d\mathbb{P})(\omega').$$

Intuitively, ∇V represents the first-order derivative of the functional V with respect to the distribution $\mathbb{P}$, while $\nabla^2 V$ represents its second-order derivative. They extend the classical notions of first and second derivatives to functionals defined on the space of probability measures. To apply Itô's lemma, we need to impose some regularity to the functional V. We use the notion of C^2 regularity introduced by Lions (see Cardaliaguet 2012):

Definition 3: *V is C^2 in the sense of Lions if it admits first and second variations and V, ∇V, and $\nabla^2 V$ are twice continuously differentiable.*

Equipped with these tools, we can now write the Hamilton Jacobi Bellman equation in the continuum of agents' case:

$$\rho V(K,\mathbb{P}) = \sup_{c,c^P,\beta,\beta^A,k} \quad \rho u(c^P) + V_K\left(\mu K - c^P - \int c(\omega)d\mathbb{P}(\omega)\right)$$

$$+\frac{1}{2}V_{KK}\sigma_A^2K^2 + \int \partial_\omega \nabla V(\omega)\rho(\omega - u(c(\omega)))d\mathbb{P}(\omega)$$

$$+\frac{1}{2}\int \partial^2_{\omega\omega}\nabla V(\omega)\left(\sigma^2\beta^2(\omega) + \sigma_A^2\beta_A^2(\omega)\right)d\mathbb{P}(\omega) + \int \partial^2_{\omega K}\nabla V(\omega)\sigma_A^2\beta_A(\omega)Kd\mathbb{P}(\omega)$$

$$+\frac{1}{2}\int\int \partial^2_{\omega\omega'}\nabla^2 V(\omega,\omega')\sigma_A^2\beta_A(\omega)\beta_A(\omega')d\mathbb{P}(\omega)d\mathbb{P}(\omega'), \qquad (25)$$

where the supremum is subject to the incentive compatibility condition and the capital allocation constraint.

When $\partial^2_{\omega\omega}\nabla V(\omega, K) < 0$ (which will be checked ex post), it is optimal to bind the incentive compatibility condition[13] (18). This determines the control $\beta(\omega)$:

$$\beta(\omega) = \rho k(\omega)u'(c(\omega)) > 0. \qquad (26)$$

Because of risk aversion, constrained optimality implies that the sensitivity of each agent's continuation utility to its current performance is exactly what is needed for incentive compatibility. Thus, (26) enables us to eliminate control $\beta(.)$.

Denoting by λ the Lagrange multiplier associated with the capital allocation constraint, we thus obtain the final expression of the HJB equation:

$$\rho V(K,\mathbb{P}) = \sup_{c,c^P,\beta^A,k} \quad \rho u(c^P) + \left(\mu K - c^P - \int c(\omega)d\mathbb{P}(\omega)\right)V_K + \frac{\sigma_A^2K^2}{2}V_{KK}$$

$$+\int[\rho(\omega - u(c(\omega)))\partial_\omega\nabla V(\omega) + \frac{(\sigma\rho k u'(c))^2(\omega) + (\sigma_A\beta_A)^2(\omega)}{2}\partial^2_{\omega\omega}\nabla V(\omega)]d\mathbb{P}(\omega)$$

[13] For the same reason, it would be suboptimal to make ω_t^i depend on the idiosyncratic shocks of other agents.

$$+\int[\partial_{\omega K}\nabla V(\omega)\sigma_A^2\beta_A(\omega)K-\lambda(k(\omega)-K)]d\mathbb{P}(\omega)$$

$$+\frac{\sigma_A^2}{2}\int\int\partial_{\omega\omega'}\nabla^2V(\omega,\omega')\beta_A(\omega)\beta_A(\omega')d\mathbb{P}(\omega)d\mathbb{P}(\omega'). \tag{27}$$

We close this section by stating the verification theorem, proved in appendix:

Proposition 3 *If V is a solution of the Hamilton Jacobi Bellman equation (27) that is C^2 in the sense of Lions and satisfies the transversality condition $\lim_{t\to\infty}e^{-\rho t}\mathbb{E}[V(K_t,\mathbb{P}_t)]=0$, it is the optimal value function of the principal in the asymmetric information case.*

5.4 An explicit solution when utility is CRRA

By analogy with the first best, we guess (and then verify) that for CRRA utility functions the value function of the principal only depends on two scalars, aggregate capital K_t and an exhaustive statistics for $\mathbb{P}_t$, namely average inverse utility:

$$A_t=\int u^{-1}(\omega)d\mathbb{P}_t(\omega). \tag{28}$$

The value function and the optimal controls then have the simple form stated in the following proposition:

Proposition 4 *When u is CRRA and $\rho>(1-\alpha)(\mu-\alpha\frac{\sigma_A^2}{2})$, the constrained optimal mechanism exists. It is parametrized by four constants (a,γ,y,φ) and has the following features:*

- *The inverse utility of each agent is proportional to their capital:*

$$u^{-1}(\omega_t)=ak(\omega_t), \tag{29}$$

where the constant a is defined as $\frac{A_0}{K_0}$.

- *Average inverse utility is proportional to aggregate capital:*

$$A_t = \int u^{-1}(\omega) d\mathbb{P}_t(\omega) = aK_t. \tag{30}$$

- *Aggregate capital and individual capital follow different geometric Brownian motions:*

$$\frac{dK_t}{K_t} = (\mu - \gamma - \frac{\rho}{y})dt + \sigma_A dZ_t^A, \tag{31}$$

  and

$$\frac{dk(\omega_t)}{k(\omega_t)} = \frac{dK_t}{K_t} + \varphi\sigma dZ_t. \tag{32}$$

- *The consumption flow of the principal is a constant fraction* γ *of aggregate capital*

$$c_t^P = \gamma K_t. \tag{33}$$

- *The consumption flow of each agent is proportional to the capital under their management:*

$$c(\omega_t) = \frac{\rho k(\omega_t)}{y}. \tag{34}$$

- *The value function of the principal is a function* v *of aggregate capital* K *and average inverse utility* A*:*

$$V(K_t, \mathbb{P}_t) = v(K_t, A_t). \tag{35}$$

Like in the first best, the value function of the principal $V(K, \mathbb{P})$ depends on the distribution of utilities $\mathbb{P}$ only via the average inverse utility A. Moreover, along the optimal path, the ratio of average inverse utility over capital $\frac{A_t}{K_t}$ remains equal to the constant $a = \frac{A_0}{K_0}$. This constant a is the initial aggregate utility of the agents (per unit of initial capital), which they must be offered to match their outside option. Thus, a can be interpreted as a measure of the bargaining power of the agents.

Equation (33) states that the consumption of the principal is a constant fraction of capital and equation (31) states that aggregate capital follows a geometric Brownian motion. These functional forms are similar to their first best counterparts, although the parameters are different under symmetric and asymmetric information. On the other hand, in contrast with the first best, the consumption and capital of each agent reflect individual performance. In the first best, capital allocation was driven by diversification motives: This implied that capital was equally split across agents. In contrast, with asymmetric information, capital allocation is driven by incentives motives: This implies, as can be seen in equation (32), that capital growth rate for agents with unexpectedly good (resp. bad) performance is larger (resp. lower) than aggregate capital growth rate. This can be interpreted as a capital allocation rule that rewards performance relative to the market benchmark, and punishes underperformance relative to that benchmark.

The proof of Proposition 4 is established in the next subsection for the logarithmic case using a guess and verify approach, in Propositions 5 and 6. The analysis of the power utility case is in the appendix.

6 The Logarithmic Utility Case

This section focuses on the log-utility case. We first establish Proposition 4 in this case, by taking the guess-and-verify approach. Then we study the properties of second-best allocations and characterize the information-constrained Pareto frontier.

6.0.1 Guess

Our guess is that consumption and capital are as stated in Proposition 4, i.e. the consumption of the principal is as given in (33), the consumption of the agent is as given in (34), the agent capital is given by (32) and (29), and $\varphi = y$. This implies that total consumption at date t equals $(\gamma + \frac{\rho}{y})K_t$, which, combined with (31), implies

$$d(\log K_t) = (\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2}{2})dt + \sigma_A dZ_t^A. \tag{36}$$

Integrating this expression from 0 to t gives

$$\log K_t = \log K + (\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A. \tag{37}$$

6.0.2 Solving for the value function under the guess

We now characterize the solution of the problem under our conjecture. The conjecture implies that the principal's utility $V(K, \mathbb{P})$ satisfies

$$V(K, \mathbb{P}) = \mathbb{E}[\int_0^\infty \rho e^{-\rho t} \log(\gamma K_t) dt] = \log(\gamma K) + \frac{1}{\rho}(\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2}{2}).$$

Similarly, by the participation constraint, an agent's initial utility ω satisfies

$$\omega = \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \log(\frac{\rho k_t}{y}) dt\right].$$

By integrating equation (32) we obtain

$$\log k_t = \log k(\omega) + (\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2})t + \sigma_A Z_t^A + \sigma y Z_t.$$

Consequently:

$$\omega = \log \frac{\rho k(\omega)}{y} + \frac{\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2}}{\rho}. \tag{38}$$

Note that the parameter y influences agents' continuation utilities in two ways: their consumption per unit of capital is $\frac{\rho}{y}$ and the sensitivity of the capital allocation to performance is y. For logarithmic utility, equation (29) in Proposition 4 writes as

$$k(\omega) = \frac{\exp \omega}{a}. \tag{39}$$

Thus, (38) is equivalent to

$$\log \frac{ay}{\rho} = \frac{\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2}}{\rho}, \tag{40}$$

Under the conjecture, the value function of the principal is thus

$$V(K, \mathbb{P}) = \log K + v(a)$$

where

$$v(a) = \sup_{\gamma, y} \Big(\log \gamma + \frac{1}{\rho}(\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2}{2}) \Big), \tag{41}$$

under constraint (40). This constraint is compatible with $\gamma > 0$ only if

$$a < \frac{\rho}{y} \exp \left[\frac{\mu - \frac{\sigma_A^2}{2} - \frac{\rho}{y} - \frac{\sigma^2 y^2}{2}}{\rho} \right].$$

Thus, the principal's problem only has a solution when a is less than some value defined by

$$a_{max} = \max_y \frac{\rho}{y} \exp \left[\frac{\mu - \frac{\sigma_A^2}{2} - \frac{\rho}{y} - \frac{\sigma^2 y^2}{2}}{\rho} \right].$$

Building on the above analysis, we obtain the following proposition.

Proposition 5 *Under our conjecture, with logarithmic utility the value function of the principal is* $V(K, \mathbb{P}) = \log K + v(a)$*, where* $v(a)$ *is defined for* $0 < a < a_{max}$ *by*

$$v(a) = \sup_{\gamma,y} \Big(\log \gamma + \frac{1}{\rho}(\mu - \frac{\sigma_A^2}{2} - \gamma - \frac{\rho}{y}) \Big), \tag{42}$$

$$s.t. \log \frac{ay}{\rho} = \frac{\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2}}{\rho}.$$

The solution of this problem is such that

$$\gamma = \rho - \frac{\rho}{y + \frac{\sigma^2}{\rho} y^3}, \tag{43}$$

where $y = y(a)$ *is defined implicitly by*

$$a = \frac{\rho}{y} \exp\left[\frac{\mu - \rho}{\rho} - \frac{\sigma^2 y}{\rho + \sigma^2 y^2} - \frac{\sigma^2 y^2 + \sigma_A^2}{2\rho} \right]. \tag{44}$$

The function v is not explicit, except when $\sigma = 0$ in which $v(a) = \log(m - a)$, corresponding to the first best value function characterized in Proposition 3.

6.0.3 Verifying the guess

Having characterized the value function under our conjecture, we now establish that this function satisfies the Hamilton-Jacobi-Bellman equation.

Proposition 6 *The value function* V *in Proposition 5 satisfies the Hamilton-Jacobi-Bellman equation and the transversality condition. Therefore, it is the solution of the control problem of the principal.*

The proof of the proposition is in the appendix.

6.0.4 Properties of second best allocations

This subsection offers an economic discussion of the properties of the above characterized optimal mechanism.

First, we consider the optimal exposure of agents to aggregate risk. To do so, we take the first order condition with respect to $\beta_A(\omega)$ of the value function in Proposition 5. This yields our next proposition:

Proposition 7 *With logarithmic utility, in the optimal contract, the agent sensitivity to aggregate risk is* $\beta_A \equiv 1$.

The value of β_A is the same in the second-best and first-best. This is because aggregate risk is publicly observable and unaffected by agents. Therefore, it can be shared optimally without jeopardizing incentives.

Second, we turn to the optimal exposure of agents to their idiosyncratic risk, which is given by the constant y. The right-hand side of equation (44) decreases in y. This yields our next proposition:

Proposition 8 *The exposure of agents to their idiosyncratic risk, y, decreases with their bargaining power, a.*

The interpretation of the proposition is the following. As the bargaining power of the agents, a, increases, the agents can demand larger utility, which the principal grants them by reducing their costly risk exposure, which, however, reduces the principal's utility. The proposition 8 is illustrated in Figure 1, which plots y as a function of a.[14] To generate the figure, the expected productivity of the risky production technology, μ, is set at 15%, while the standard deviation of the idiosyncratic risk, σ, is set at 10%. Moreover, the agent discount rate, ρ, is set to 4%, while the standard deviation of the aggregate shock, σ_A, is set to 5%.

[14] As can be seen in equation (44), when a goes to 0, y must go to infinity

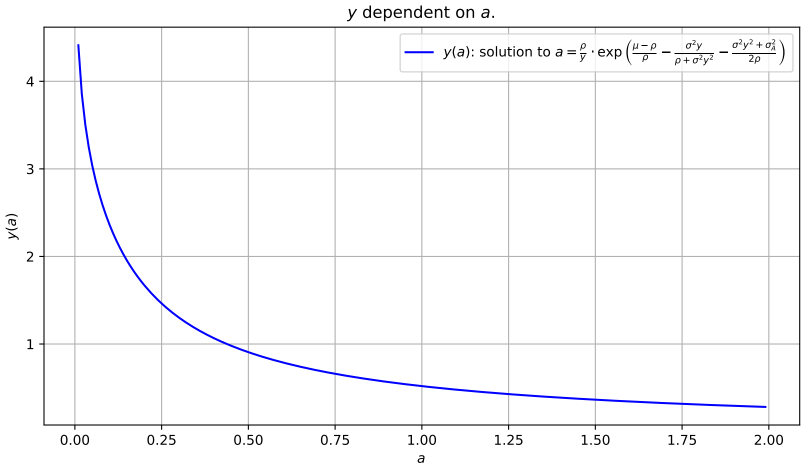


Figure 1: Idiosyncratic risk exposure as a function of bargaining power.

Third, we examine the dynamics of capital and continuation utilities in the optimal mechanism. Propositions 4 and 5 yield the following proposition:

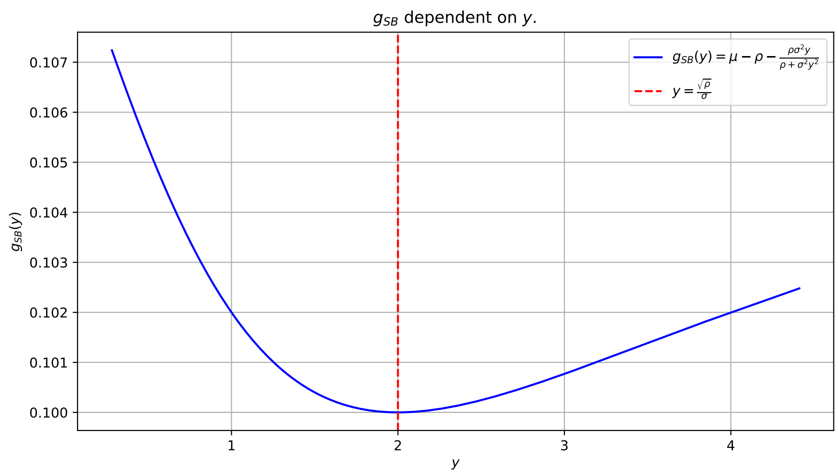
Proposition 9 *Second best optimal allocations can be parameterized by y. They are such that, at each date t aggregate capital grows in expectation at the constant rate*

$$g_{SB} = \mu - \rho - \frac{\rho\sigma^2 y}{\rho + \sigma^2 y^2}. \tag{45}$$

and agents' continuation utilities follow a drifted Brownian motion:

$$\omega_t = \omega + \left(g_{SB} - \frac{y^2\sigma^2 + \sigma_A^2}{2}\right) t + y\sigma Z_t + \sigma_A Z_t^A. \tag{46}$$

Equation (45) shows that when $\sigma = 0$, so that there is no incentive problem, the growth rate is equal to its first best level $(\mu - \rho)$. However, when $\sigma > 0$, the expected growth is lower in the second best than in first best. This is driven by the fact that incentive compatibility precludes perfect risk-sharing. To relax the incentive constraint, and thus improve risk-sharing, it



Parameters: $\mu = 0.15$, $\rho = 0.04$, $\sigma = 0.1$, $\sigma_A = 0.05$.

Figure 2: Expected growth as a function of idiosyncratic risk exposure.

is optimal to lower capital and raise consumption relative to the first best. This, in turn, reduces growth.

Equation (45) also implies that the aggregate expected growth rate g_{SB} is decreasing for $y < \sqrt{\rho}/\sigma$, and increasing for $y \geq \sqrt{\rho}/\sigma$. This is illustrated in Figure 2, which is generated for the same parameter values as in Figure 1. The figure shows that the expected growth is nonmonotonic in y. This is because an increase in y has two effects that go in opposite directions: On the one hand, as can be seen in (34), other things equal an increase in y reduces agents' consumption, which increases growth. On the other hand, as can be seen in Equation (43), an increase in y increases principal's consumption, which reduces growth. As can be seen in the figure, for large values of y, the former effect dominates the latter.

Equation (46), illustrated in Figure 3, characterizes the dynamics of the continuation utility of the agents. Figure 3 is generated for the same parameter values as Figure 1 and Figure 2, except that in Figure 1 and Figure 2, a varied between 0 and 2 while in Figure 3, a is constant and set to 1.

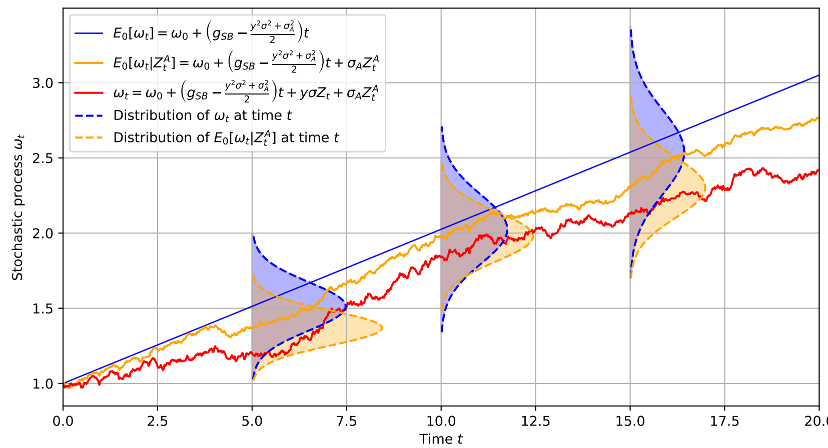


Figure 3: Dynamics of agents' continuation utility.

The blue line in Figure 3 is the expectation of the time t continuation utility of an agent conditional on the information of time 0. This expectation is linear in t, since it is equal to

$$\omega + (g_{SB} - \frac{y^2\sigma^2 + \sigma_A^2}{2})t.$$

The slope reflects the opposite effects of expected aggregate growth, which tends to increase the utility of agents, and of costly risk bearing, which tends to reduce the utility of agents. If capital productivity μ is large enough, the positive effect of growth dominates the negative effect of costly risk-bearing, so that, in expectation, agent's utility tends to increase. This is the case for the parameters used to generate the figure. So, in the figure, the blue line is increasing.

The blue bell shape in the figure is the distribution of agents' time t continuation utility around its expectation conditional on time 0 information. The dispersion around the expectation reflects the weighted sum of the two

Brownian terms

$$y\sigma Z_t + \sigma_A Z_t^A.$$

The standard deviation of this Brownian term increases with (the square root of) time. Consequently, in Figure 3, the width of the blue bell shape increases with time, i.e., the inequality between agents increases with time. This increase in inequality is due to the incentive compatibility constraint, which implies that agents' continuation utilities must react to their idiosyncratic performance.

Similarly to the first best, if the capital productivity (μ) is high enough, so that the drift in ω_t is positive, all agents' utilities go to infinity.[15] The difference between the first and second best is that, in the latter, the productivity needed for the continuation utilities to go to infinity is greater, to compensate for the negative effect of exposure to the idiosyncratic risk.

The stochastic process in orange in the figure is the expected continuation utility of the agent at time t, conditional on one sample path for the realization of the aggregate shock Z_t^A, that is,

$$\omega + (g_{SB} - \frac{y^2\sigma^2 + \sigma_A^2}{2})t + \sigma_A Z_t^A.$$

The orange bell shape is the distribution of the continuation utility at time t around this conditional expectation. The dispersion around this conditional expectation reflects only the idiosyncratic risk, multiplied by the exposure of the agent to this risk, that is,

$$y\sigma Z_t.$$

[15]Intuitively, this is due to the fact that the expectation of the continuation utility increases linearly with time, while the standard deviation increases as the square root of time. When t goes to infinity, the linear increase dominates the square root increase.

The blue bell shape shows more dispersion than the orange bell shape, since the former reflects uncertainty about Z_A^t and Z_t, while the latter only reflects uncertainty about Z_t. However, similar to the blue bell shape, the width of the orange bell shape, increases with time, reflecting the increasing cross-sectional inequality between agents.

Finally, the stochastic process in red is the realization of the stochastic process of an agent's continuation utility (ω_t), conditional on one sample path for aggregate risk (Z_A^t), and one sample path for the agent's idiosyncratic risk (Z_t). Thus, Figure 3 illustrates that there is a simple relation between the continuation utility of an agent at date t and its performance over $(0, t)$. The total productivity of the agent on $(0, t)$ is $\mu t + \sigma Z_t + \sigma^A Z_t^A$. When there is no aggregate risk, optimal compensation implies a simple, affine relation between continuation utility ω_t and this performance measure, similarly to Holmström and Milgrom (1987). With aggregate risk, optimal compensation is more complex as it must also take into account aggregate performance.

6.0.5 Information constrained Pareto frontier

The above analysis yields a characterization of the information-constrained Pareto frontier in the space of inverse utility. Substituting the expression of γ from (43) into (46), and (41), we obtain

$$\exp \omega = \frac{\rho}{y} k(\omega) \exp \left[\frac{g_{SB}}{\rho} - \frac{\sigma^2 y^2 + \sigma_A^2}{2\rho} \right], \tag{47}$$

where g_{SB} is the growth rate given in Proposition 9, and

$$\exp V(K, \mathbb{P}) = \rho \left(1 - \frac{1}{y + \frac{\sigma^2}{\rho} y^3} \right) K \exp \left[\frac{g_{SB} - \frac{\sigma_A^2}{2}}{\rho} \right]. \tag{48}$$

Equation (47) reflects that each agent consumes a fraction $\frac{\rho}{y}$ of its capital under management $k(\omega)$, which grows at the average rate g_{SB}, with volatility $\sigma^2 y^2 + \sigma_A^2$ generating a risk premium and is discounted at rate ρ. Similarly (48) reflects that the principal consumes a fraction $\gamma = \rho\left(1 - \frac{1}{y + \frac{\sigma^2}{\rho} y^3}\right)$ of aggregate capital with volatility σ_A^2. For the principal, the risk premium only reflects the aggregate risk, while for the agent, it also reflects the idiosyncratic risk which the agent must bear for incentive reasons.

Figure 4 depicts the Pareto frontier generated by equations (47) and (48). In general, the Pareto frontier depends on the utilities of all agents. In order to draw a figure in two dimensions, we consider the case in which all agents start with one unit of capital, and thus all have the same initial utility. Thus, the horizontal axis is the representative agent's lifetime discounted expected utility, while the vertical axis is the principal's lifetime discounted expected utility. Different points on the Pareto frontier correspond to different values of y. As y increases, we move along the Pareto frontier to the northwest, corresponding to an allocation that is more favorable to the principal and less favorable to the agent. The figure also depicts the Pareto frontier for different values of σ. This shows that as the idiosyncratic risk increases, making the moral hazard problem more severe, the Pareto frontier shifts downwards.

7 Conclusion

This paper studies dynamic capital allocation and risk sharing under information asymmetry. The principal, for example, a pension fund, sovereign fund, or family office, allocates capital among multiple agents, for example, private equity, real estate, or infrastructure fund managers. The reason why the principal delegates the management of the assets, i.e., agents' better understanding of the assets, is also the reason why there are information asymmetries

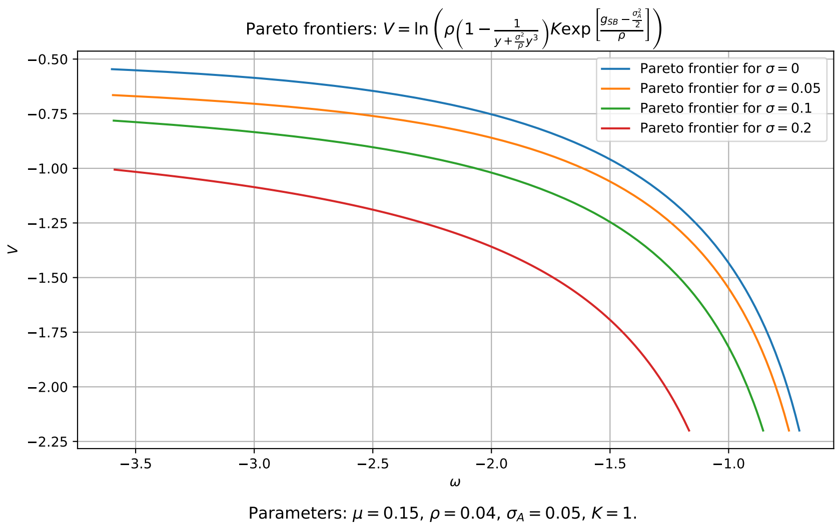


Figure 4: Pareto frontiers.

between the principal and the agents. Thus, we adopt a mechanism design approach to study incentive-constrained dynamic capital allocation and risk sharing between the agents and the principal.

Without incentive constraints, capital allocation would be driven by risk and return considerations only, as in Merton (1969). In contrast, in our setting, capital allocation among agents is driven by incentive considerations, so our analysis can be understood as the generalization of the Merton (1969) approach to the incentive-constrained case. We show that the optimal mechanism allocates more capital to agents with better performance. In addition, agents are compensated with fees that are proportional to the amount of assets under their management. Thus, our theoretical results are in line with stylized facts from the fund management industry.

In second-best optimal allocations, agents are partially exposed to the idiosyncratic risk of the assets they manage. This makes risk-sharing imperfect relative to the first-best. This increases the cost of holding risky capital and tilts the consumption-investment trade-off towards less investment. So, there is less capital accumulation than in the first best.

To obtain these results, we solve an optimal control problem in which one of the state variables (the distribution of agents' utilities) is infinite-dimensional. Developing this new methodology is one of the contributions of our analysis and could be used to study other economic problems with endogenous heterogeneity across agents.

8 Appendix

The appendix follows the outline in the main text, which describes the guess-and-verify procedure in both the symmetric information and the asymmetric information cases. The appendix is split into two parts, corresponding to the symmetric and asymmetric information cases, respectively. For each part, we construct the guessed value function and then verify it satisfies the optimality conditions. Within each part, we first solve the problem with logarithmic utility and then with power utility.

8.1 The symmetric information case

Our first step is to prove Proposition 2 for the logarithmic utility case.

8.1.1 Solving for the value function under the guess

Substituting (14) in the resource constraint (3), yields the dynamics of capital:

$$\frac{dK_t}{K_t} = \left(\mu - \gamma^P - \frac{1}{N}\sum_i \gamma^i\right) dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_i dZ_t^i. \tag{49}$$

Thus K_t follows a log-normal distribution and

$$\mathbb{E}(\log K_t) = \log K + (g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N})t, \tag{50}$$

where $g = \mu - \gamma^P - \frac{1}{N}\sum_i \gamma^i$. An integration gives the principal's value function in the case of logarithmic utility

$$V(K, \mathbb{W}) = \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \log(\gamma^P K_t) dt\right] = \log(\gamma^P K) + \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right). \tag{51}$$

The continuation pay-off of agent i with logarithmic utility is computed along the same lines:

$$\omega_t^i = \log(\gamma^i K_t) + \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right), \tag{52}$$

implying that the ratio $\frac{\exp \omega_t^i}{K_t}$ remains constant along optimal trajectories. This property implies in turn that risk is equally shared across agents. Indeed, by differentiation of (52) we obtain:

$$d\omega_t^i = d(\log K_t) = (g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N})dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_i dZ_t^i, \tag{53}$$

and thus

$$\beta^{A,i} \equiv \beta^{ij} \equiv 1, i, j = 1, ..., N. \tag{54}$$

Equation (54) shows that, when information is symmetric, idiosyncratic risks as well as aggregate risk are equally shared among all participants, to reap the gains from mutualization.

Taking the exponential of (51) and (52) and summing them, the unconstrained Pareto frontier for logarithmic utilities can be parameterized as follows:

$$\exp V(K, \mathbb{W}) + \frac{\sum_i \exp \omega^i}{N} = K \exp \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right)(\mu - g). \tag{55}$$

Maximizing the right-hand side over g gives:

$$\mu - g = \rho.$$

Thus in the optimal mechanism, the aggregate consumption rate is equal to the discount rate:

$$\gamma^P + \frac{1}{N}\sum_i \gamma^i = \rho, \tag{56}$$

while the expected growth rate is $g_{FB} = \mu - \rho$. So, the maximum of the right-hand side of (55) is

$$mK \equiv \rho \exp \frac{1}{\rho}\left(\mu - \rho - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right) K,$$

which determines the constant m. Finally, the value function of the principal can be obtained by solving for V in the equation of the Pareto frontier.

Proposition 10 *For logarithmic utility, under our conjecture, the value function of the principal is:*

$$V(K, \mathbb{W}) = \log K + \log\left(m - \frac{A}{K}\right), \tag{57}$$

where

$$m = \rho \exp \frac{1}{\rho}\left(\mu - \rho - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right). \tag{58}$$

Proof of Proposition 10 Consider the function θ defined on the interval $(-\infty, \mu)$ by

$$\theta(g) = \exp \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right)(\mu - g).$$

The function θ appears on the right-hand side of equation (55). Observe that θ is non-negative continuous with $\lim_{g \to -\infty} \theta(g) = \theta(\mu) = 0$. Thus, θ admits a maximum at a critical point. However, it is easy to verify that the function θ has a unique critical point $g_{FB} = \mu - \rho$. Moreover, the maximum value of θ is

$$m = \theta(\mu - \rho) = \rho \exp \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right).$$

Finally, by taking the logarithm in (55), we obtain the guessed value function (57).

The value function in (57) is the same as the value function in (15) specialized to the case of logarithmic utility. Note, however, that the value function is only defined when $mK > A$, i.e., when there is a minimum amount of capital needed to provide agents with the utility vector $\mathbb{W}$.

QED

8.1.2 Verifying the guess

The next proposition completes our guess and verify approach by establishing that the solution obtained under our guess, and stated in equations (57) and (58), is a solution to the HJB equation.

Proposition 11 *The value function stated in equations (57) and (58) satisfies the HJB equation and the transversality condition. So, the solution to the principal's problem is the one that was guessed in Proposition 2.*

Proof First recall that the ratio $\frac{A_t}{K_t}$ is a constant a and thus the value function V can be written

$$V(K, \mathbb{W}) = \log K + \log(m - a).$$

Its partial derivatives are

$$V_K = \frac{m}{K(m-a)}, V_{KK} = -\frac{m^2}{K^2(m-a)^2}, V_{\omega_i} = -\frac{\exp \omega_i}{NK(m-a)},$$

$$V_{\omega_i K} = \frac{m \exp \omega_i}{NK^2(m-a)^2}, V_{\omega_i \omega_i} = -\frac{\exp 2\omega_i}{N^2 K^2 (m-a)^2} - \frac{\exp \omega_i}{NK(m-a)},$$

and, for $j \neq i$,

$$V_{\omega_i \omega_j} = -\frac{\exp \omega_i \exp \omega_j}{N^2 K^2 (m-a)^2}.$$

We plug the conjectured value function in (8) to obtain, denoting by π the controls $(k, c^P, c, \beta, \beta_A)$ to alleviate notations,

$$
\begin{aligned}
\rho \log K + \rho \log(m-a) = \sup_{\pi} \Bigg\{ & \rho \log(c^P) - \sum_{i=1}^{N} \rho(\omega_i - \log(c^i)) \frac{\exp \omega_i}{NK(m-a)} \\
& + \frac{m}{K(m-a)} \Big(\mu K - c^P - \frac{1}{N} \sum_{i=1}^{N} c^i \Big) \\
& + \sum_{i=1}^{N} \Big(\sigma_A^2 K \beta^{A,i} + \frac{\sigma^2}{N^2} \sum_{j=1}^{N} k^j \beta^{ij} \Big) \frac{m \exp \omega_i}{NK^2(m-a)^2} \\
& - \frac{1}{2} \Big(\sigma_A^2 K^2 + \frac{\sigma^2}{N^2} \sum_{j=1}^{N} (k^j)^2 \Big) \frac{m^2}{K^2(m-a)^2} \\
& - \frac{1}{2} \sum_{j \neq i} \Big(\sigma_A^2 \beta^{A,i} \beta^{A,j} + \frac{\sigma^2}{N^2} \sum_{l=1}^{N} \beta^{il} \beta^{jl} \Big) \frac{\exp \omega_i \exp \omega_j}{N^2 K^2 (m-a)^2} \\
& - \frac{1}{2} \sum_{i=1}^{N} \Big(\sigma_A^2 (\beta^{A,i})^2 + \frac{\sigma^2}{N^2} \sum_{l=1}^{N} (\beta^{il})^2 \Big) \Big(\frac{\exp 2\omega_i}{N^2 K^2 (m-a)^2} + \frac{\exp \omega_i}{NK(m-a)} \Big) \\
& + \lambda \Big(\sum_{i=1}^{N} k^i - K \Big) \Bigg\}
\end{aligned}
$$

The first order conditions give:

- $\frac{\rho}{c^P} = \frac{m}{K(m-a)}$ implying $c^P = \rho(1 - \frac{a}{m})K$,
- $\frac{\rho}{c^i} \frac{\exp \omega_i}{NK(m-a)} = \frac{m}{NK(m-a)}$ implying $c^i = \frac{\rho \exp \omega^i}{m}$,
- $\beta^{ij} = \beta^{A,i} = 1$ for all i, j
- $k^i = K$, for all i.

Equation (8) rewrites

$$\begin{aligned}
\rho\log(K)+\rho\log(m-a) &= \rho\log(\frac{\rho}{m})+\rho\log(m-a)+\rho\log(K)\\
&+\sum_{i=1}^{N}\rho\log(\frac{\rho}{m})\frac{\exp\omega_i}{NK(m-a)}\\
&+\frac{m}{K(m-a)}\Big(\mu K-\rho(1-\frac{a}{m})K-\frac{\rho}{m}aK)\Big)\\
&+\sum_{i=1}^{N}\big(\sigma_A^2K+\frac{\sigma^2}{N}K\big)\frac{m\exp\omega_i}{NK^2(m-a)^2}\\
&-\frac{1}{2}\big(\sigma_A^2K^2+\frac{\sigma^2}{N}K^2\big)\frac{m^2}{K^2(m-a)^2}\\
&-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\big(\sigma_A^2+\frac{\sigma^2}{N}\big)\frac{\exp\omega_i\exp\omega_j}{N^2K^2(m-a)^2}\\
&-\frac{1}{2}\sum_{i=1}^{N}\big(\sigma_A^2+\frac{\sigma^2}{N}\big)\frac{\exp\omega_i}{NK(m-a)}.
\end{aligned}$$

After simplifications, we obtain

$$0=\frac{m}{m-a}\Big(\rho\log(\frac{\rho}{m})+\mu-\rho-\frac{1}{2}\big(\sigma_A^2+\frac{\sigma^2}{N}\big)\Big),$$

which is true by the definition of m given in (58).

It remains to prove the transversality condition. For $\varepsilon>0$, let us define the set

$$\mathcal{A}_\varepsilon=\{\pi\text{ such that }A_t^{(\pi)}\leq(m-\varepsilon)K_t^{(\pi)}\text{ for }t\geq 0\},$$

where A_t^π and K_t^π are respectively the average inverse utility and the capital processes under the control policy π and the set $\mathcal{A}$ as the union of $\mathcal{A}_\varepsilon$. We will prove that for every control $\pi\in\mathcal{A}$, we have the transversality condition

$\lim_{t\to+\infty} e^{-\rho t}\left(\log K_t^\pi + \log(m - a_t^\pi)\right) = 0$ where

$$a_t^\pi = \frac{A_t^{(\pi)}}{K_t^{(\pi)}}.$$

Take $\pi \in \mathcal{A}$, then there is $\varepsilon > 0$, such that $\pi \in \mathcal{A}_\varepsilon$ and we have $\log(m - a_t^\pi) \leq \log(\frac{1}{\varepsilon})$. On the other hand, because the consumption processes are non-negative, the capital process $K^{(\pi)}$ given in (3) is dominated by the Geometric Brownian motion

$$K \exp[(\mu - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A].$$

Consequently, we have

$$\mathbb{E}\left[e^{-\rho t}\log(K_t)\right] \leq \mathbb{E}\left[e^{-\rho t}(\log(K) + (\mu - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A)\right] \text{ which converges to 0.}$$

QED

8.1.3 The power utility case

We now prove Proposition 2 in the power utility case.

Let $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$. Proceeding in a similar manner to Proposition 10, we start by computing the guessed value function. The dynamics of capital is the same as in the log utility case:

$$\frac{dK_t}{K_t} = \left(\mu - \gamma^P - \frac{1}{N}\sum_i \gamma^i\right)dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_i dZ_t^i. \qquad (59)$$

Thus K_t follows a log-normal distribution and

$$\log K_t = \log K + (g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N})t + \sigma_A Z_t^A + \frac{\sigma}{N}\sum_i Z_t^i, \qquad (60)$$

where $g = \mu - \gamma^P - \frac{1}{N}\sum_i \gamma^i$. Thus

$$\mathbb{E}\left[u(\gamma^P K_t)\right] = u(\gamma^P K)\exp\left[(1-\alpha)gt - \alpha(1-\alpha)(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N})t\right]$$

Since $g < \mu$, the condition $\rho > (1-\alpha)\mu - \alpha(1-\alpha)\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)$, guarantees the convergence of the integral giving the principal's utility:

$$V(K, \mathbb{W}) = \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(\gamma^P K_t)dt\right] = \frac{\rho u(\gamma^P K)}{\rho - (1-\alpha)g + \alpha(1-\alpha)\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)}. \tag{61}$$

The continuation pay-off of agent i at date t is computed along the same lines:

$$\omega_t^i = \frac{\rho u(\gamma^i K_t)}{\rho - (1-\alpha)g + \alpha(1-\alpha)\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)}, \tag{62}$$

implying that $u^{-1}(\omega_t^i)$ is proportional to K_t along the optimal path. Specifically, we have

$$u^{-1}(\omega_t^i) = \gamma^i \left[\frac{\rho}{\rho - (1-\alpha)g + \alpha(1-\alpha)\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)})\right]^{\frac{1}{1-\alpha}} K_t = \gamma^i \theta K_t,$$

where

$$\theta = \left[\frac{\rho}{\rho - (1-\alpha)g + \alpha(1-\alpha)\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)})\right]^{\frac{1}{1-\alpha}},$$

and similarly for the principal

$$u^{-1}(V(K_t, \mathbb{W}_t)) = \gamma^P \left[\frac{\rho}{\rho - (1-\alpha)g + \alpha(1-\alpha)\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)})\right]^{\frac{1}{1-\alpha}} K_t = \gamma^P \theta K.$$

Since $\gamma^P + \frac{1}{N}\sum_i \gamma^i = \mu - g$, the equation of the Pareto frontier can be written

$$u^{-1}(V(K, \mathbb{W})) + \frac{\sum_i u^{-1}(\omega^i)}{N} = mK,$$

where we have dropped the time index because the problem is stationary, and the constant m is defined as

$$m = \sup_g \left((\mu - g) \left[\frac{\rho}{\rho - (1-\alpha)g + \alpha(1-\alpha)\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)})\right]^{\frac{1}{1-\alpha}} \right)$$

It is easy to see that the supremum is obtained for

$$g_{FB} = \frac{1}{\alpha}\left[\mu - \rho - (1-\alpha)\alpha\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N}\right)\right],$$

and the constant m is thus equal to

$$m = \rho\left[1 - \frac{(1-\alpha)}{\alpha\rho}\left(\mu - \rho - \alpha(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2N})\right)\right]^{\frac{-\alpha}{1-\alpha}}. \tag{63}$$

It remains to prove that the function $V(K, \mathbb{W})$ given in equation (15) solves the HJB equation. To see this, we set

$$a_t = \frac{\sum_{i=1}^N u^{-1}(\omega_t^i)}{NK_t},$$

to write $V(K, \mathbb{W}) = u(K)(m-a)^{1-\alpha}$, where according to the Pareto frontier equation, a has to be strictly lower than m in order to have a finite value for V. We have

$$
\begin{aligned}
\rho u(K)(m-a)^{1-\alpha} = \sup_{\pi} \Bigg\{ & \rho u(c^P) - \sum_{i=1}^{N} \rho(\omega_i - u(c^i)) \frac{K^{-\alpha}}{N}(m-a)^{-\alpha}\big((1-\alpha)\omega^i\big)^{\frac{\alpha}{1-\alpha}} \\
& + \Big(\mu K - c^P - \frac{1}{N}\sum_{i=1}^{N} c^i\Big) m K^{-\alpha}(m-a)^{-\alpha} \\
& + \sum_{i=1}^{N} \Big(\sigma_A^2 K \beta^{A,i} + \frac{\sigma^2}{N^2}\sum_{j=1}^{N} k^j \beta^{ij}\Big) \frac{\alpha m}{N} K^{-(1+\alpha)}(m-a)^{-(1+\alpha)}\big((1-\alpha)\omega^i\big)^{\frac{\alpha}{1-\alpha}} \\
& - \frac{1}{2}\Big(\sigma_A^2 K^2 + \frac{\sigma^2}{N^2}\sum_{j=1}^{N}(k^j)^2\Big)\alpha m^2 K^{-(1+\alpha)}(m-a)^{-(1+\alpha)} \\
& - \frac{1}{2}\sum_{j\neq i}\Big(\sigma_A^2 \beta^{A,i}\beta^{A,j} + \frac{\sigma^2}{N^2}\sum_{l=1}^{N}\beta^{il}\beta^{jl}\Big)\frac{\alpha}{N^2}\big((1-\alpha)\omega^i\big)^{\frac{\alpha}{1-\alpha}}\big((1-\alpha)w^j\big)^{\frac{\alpha}{1-\alpha}} \\
& \times K^{-(1+\alpha)}(m-a)^{-(1+\alpha)} \\
& - \frac{1}{2}\sum_{i=1}^{N}\Big(\sigma_A^2(\beta^{A,i})^2 + \frac{\sigma^2}{N^2}\sum_{l=1}^{N}(\beta^{il})^2\Big)\frac{\alpha}{N}\big((1-\alpha)\omega^i\big)^{\frac{2\alpha-1}{1-\alpha}} \\
& \times K^{-(1+\alpha)}(m-a)^{-(1+\alpha)}\big(\frac{(1-\alpha)\omega_i}{N} + K(m-a)\big) \\
& + \lambda\Big(\sum_{i=1}^{N} k^i - K\Big)\Bigg\}
\end{aligned}
$$

Proceeding analogously as in the logarithmic case, the first order conditions give:

- $c^P = \big(\frac{m}{\rho}\big)^{-\frac{1}{\alpha}}(m-a)K = \gamma K$,
- $c^i = \big(\frac{m}{\rho}\big)^{-\frac{1}{\alpha}}\big((1-\alpha)\omega^i\big)^{\frac{1}{1-\alpha}}$,
- $\beta^{ij} = (1-\alpha)w_i$ for all j,
- $\beta^{A,i} = (1-\alpha)\omega_i$ for all i
- $k^i = K$, for all i.

Equation (8) rewrites

$$\begin{aligned}\rho\frac{K^{1-\alpha}}{1-\alpha}(m-a)^{1-\alpha} = &\rho\frac{K^{1-\alpha}}{1-\alpha}(m-a)^{1-\alpha}\big(\frac{m}{\rho}\big)^{-\frac{1-\alpha}{\alpha}}\\ &-\rho\frac{1-\theta^{\alpha-1}}{1-\alpha}aK^{1-\alpha}(m-a)^{-\alpha}\\ &+gmK^{1-\alpha}(m-a)^{-\alpha}\\ &+\big(\sigma_A^2+\frac{\sigma^2}{N}\big)\alpha amK^{1-\alpha}(m-a)^{-(1+\alpha)}\\ &-\frac{1}{2}\big(\sigma_A^2+\frac{\sigma^2}{N}\big)\alpha m^2K^{1-\alpha}(m-a)^{-(1+\alpha)}\\ &-\frac{1}{2}\big(\sigma_A^2+\frac{\sigma^2}{N}\big)a^2\alpha K^{1-\alpha}(m-a)^{-(1+\alpha)}\\ &-\frac{1}{2}\big(\sigma_A^2+\frac{\sigma^2}{N}\big)aK^{1-\alpha}(m-a)^{-\alpha}\end{aligned}$$

After simplifications, we obtain

$$\begin{aligned}\rho(m-a) = \rho(m-a)\big(\frac{m}{\rho}\big)^{-\frac{1-\alpha}{\alpha}} - \rho(1-\theta^{\alpha-1})a\\ +(1-\alpha)mg-\frac{1}{2}\big(\sigma_A^2+\frac{\sigma^2}{N}\big)(\alpha(1-\alpha)+a).\end{aligned}$$

which is true by the definition of m and θ. QED

8.2 The asymmetric information case

8.2.1 Proof of Proposition 3:

Fix $\mathbb{P}\in\mathcal{P}_2(\mathbb{R})$ and a Lagrange multiplier λ. Let $\mathbb{P}_t$ be the conditional probability distribution of the random variable ω_t given the filtration generated by $(Z_t^A)_t$ when the initial probability distribution of ω_0 is $\mathbb{P}$. Let us consider some arbitrary feedback control $\pi(K_t,\mathbb{P}_t,\omega_t)$. Let denote by v^λ a smooth solution to the HJB equation (27). We apply Itô's formula to $e^{-\rho s}v^\lambda(K_s,\mathbb{P}_s)$ between $s=0$ and $s=t$ for $t>0$ to obtain

$$
\begin{aligned}
\mathbb{E}[e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t)] = v^\lambda(K, \mathbb{P}) \\
&+ \mathbb{E}\int_0^t e^{-\rho s}\left(-\rho v^\lambda(K_s,\mathbb{P}_s) + \left(\mu K_s - c^P - \int c(\omega)d\mathbb{P}_s(\omega)\right) v_K^\lambda(K_s,\mathbb{P}_s)\right) ds \\
&+ \mathbb{E}\int_0^t e^{-\rho s}\frac{\sigma_A^2 K_s^2}{2} v_{KK}^\lambda(K_s,\mathbb{P}_s)\, ds \\
&+ \mathbb{E}\int_0^t e^{-\rho s}\int \partial_\omega \nabla v^\lambda(\omega)(\rho(\omega - u(c(\omega))))d\mathbb{P}_s(\omega)\, ds \\
&+ \mathbb{E}\int_0^t e^{-\rho s}\int \partial_{\omega\omega}^2 \nabla v^\lambda(\omega)\frac{(\sigma\rho k u'(c))^2(\omega) + (\sigma_A\beta_A)^2(\omega)}{2} d\mathbb{P}_s(\omega)\, ds \\
&+ \mathbb{E}\int_0^t e^{-\rho s}\int \partial_{\omega K}^2 \nabla v^\lambda(\omega)\sigma_A^2 \beta_A(\omega) K d\mathbb{P}_s(\omega)\, ds \\
&+ \mathbb{E}\int_0^t e^{-\rho s}\frac{\sigma_A^2}{2}\int\int \partial_{\omega\omega'}^2 \nabla^2 v^\lambda(\omega,\omega')\beta_A(\omega)\beta_A(\omega')d\mathbb{P}_s(\omega)d\mathbb{P}_s(\omega')\, ds.
\end{aligned}
$$

We deduce from the HJB equation (27) satisfied by v^λ that

$$
\begin{aligned}
v^\lambda(K,\mathbb{P}) \geq \mathbb{E}[e^{-\rho t} v^\lambda(K_t,\mathbb{P}_t)] \\
&+ \mathbb{E}\int_0^t e^{-\rho s}\left(\rho u(c^P) + \lambda\left(K_s - \int k(\omega)d\mathbb{P}_s(\omega)\right)\right) ds.
\end{aligned}
$$

Letting t tend to $+\infty$ and using the transversality condition, we obtain

$$
\begin{aligned}
v^\lambda(K,\mathbb{P}) &\geq \mathbb{E}\int_0^\infty e^{-\rho s}\left(\rho u(c^P) + \lambda\left(K_s - \int k(\omega)d\mathbb{P}_s(\omega)\right)\right) ds. \\
&:= J_\lambda^\pi.
\end{aligned}
$$

Since the control π is arbitrary, we obtain $v^\lambda(K,\mathbb{P}) \geq V_\lambda = \sup_\pi J_\lambda^\pi$, where V_λ coincides with the principal value if the optimal controls bind the budget constraint. On the other hand, let us apply the same Itô argument with the control π_λ^* attaining the maximum in (27). We obtain

$$
v^\lambda(K,\mathbb{P}) = J_\lambda^{\pi_\lambda^*} \leq V_\lambda,
$$

which yields that $v^\lambda = V_\lambda$. This ends the proof of Proposition 3.

8.2.2 Proof of Proposition 4 in the logarithmic utility case

We show that the conjectured value function $V(K, \mathbb{P}) = \log K + v(a)$ satisfies the Bellman equation (27). To do so, we first compute the derivatives of V, using the definition $a = \frac{\int exp(\omega)d\mathbb{P}}{K}$. We have

$$\partial_K V = \frac{1}{K}(1 - av'(a)),\ \partial^2_{KK} V = -\frac{1}{K^2}\left(1 - 2av'(a) - a^2 v''(a)\right)$$
$$\partial_\omega \nabla V = \partial^2_{\omega\omega} \nabla V = \frac{\exp\omega}{K} v'(a),\ \partial^2_{\omega K} \nabla V = -\frac{\exp\omega}{K^2}(v'(a) + av''(a))$$
$$\partial^2_{\omega\omega'} \nabla V = \frac{\exp\omega\exp\omega'}{K^2} v''(a).$$

Putting in the HJB equation (27), we obtain

$$\begin{aligned}
\rho\log K + \rho v(a) = \sup_{c, c^P, \beta^A, k} & \Bigg\{\rho\log(c^P) + \left(\mu K - c^P - \int c(\omega)d\mathbb{P}(\omega)\right)\frac{1}{K}(1 - av'(a)) \\
& -\frac{\sigma_A^2}{2}\left(1 - 2av'(a) - a^2 v''(a)\right) + \int [\rho(\omega - u(c(\omega)))\frac{\exp\omega}{K} v'(a)d\mathbb{P}(\omega) \\
& -\int \sigma_A^2 \beta_A(\omega)\frac{\exp\omega}{K}(v'(a) + av''(a))d\mathbb{P}(\omega) \\
& +\int \frac{(\sigma\rho k u'(c))^2(\omega) + (\sigma_A\beta_A)^2(\omega)}{2}\frac{\exp\omega}{K} v'(a)d\mathbb{P}(\omega) \\
& \frac{\sigma_A^2}{2}\int\int \beta_A(\omega)\beta_A(\omega')d\mathbb{P}(\omega)\frac{\exp\omega\exp\omega'}{K^2} v''(a)d\mathbb{P}(\omega') \\
& -\int \lambda(k(\omega) - K)d\mathbb{P}(\omega)\Bigg\}
\end{aligned}$$

The first order condition give

$$
\begin{aligned}
&c^P = \frac{\rho}{1-av'(a)}K \text{ if } 1-av'(a)\neq 0,\\
&\sigma^2\rho^2\frac{k}{c^2}\exp\omega\frac{v'(a)}{K} = \lambda\\
&-(1-av'(a)) - \frac{\rho}{c}\exp\omega v'(a) - \sigma^2\rho^2\frac{k^2}{c^3}\exp\omega v'(a) = 0\\
&-\frac{\exp\omega}{K}(v'(a)+av''(a)) + \beta_A\frac{\exp\omega}{K}v'(a) + \frac{1}{K^2}\Big(\int\beta_A(\omega)\exp\omega d\mathbb{P}(\omega)\Big)v''(a)\exp\omega = 0.
\end{aligned}
$$

We deduce from the last line that $\beta_A(\omega)=1$ for every ω which is proved as a by product of proposition 7. The second equation gives that the ratio $\frac{k}{c^2}\exp\omega$ is constant. Using this in the third equation shows that c is proportional to $\exp\omega$ and the ratio $\frac{k}{c}$ is a constant y. In the following, we denote

$$k = \frac{1}{a}\exp\omega \text{ and } c = \frac{\rho}{ay}\exp\omega$$

Binding the capital allocation constraint gives that the ratio $\frac{A_t}{K_t}$ is the constant a which implies $c^P=\gamma K$. Finally, Equation (27) rewrites

$$\rho v(a) = \sup_{y,\gamma}\left[\rho\log\gamma + [1-av'(a)](\mu - \frac{\sigma_A^2}{2} - \gamma - \frac{\rho}{y}) + av'(a)[\rho\log(\frac{ay}{\rho}) + \frac{\sigma^2y^2}{2}]\right].$$

This coincides with the definition of v given in Proposition 5, once we have noted, by the envelope theorem, that the Lagrange multiplier of the capital allocation constraint equals $\rho av'(a)$, which proves that our guess value function satisfies HJB.

To apply the verification theorem (Proposition 3) and conclude that the function $\log(K)+v(a)$ coincides with the principal value, it remains to prove that $\mathbb{E}\left[e^{-\rho t}\left(\log(K_t)+v(\frac{A_t}{K_t})\right)\right]$ converges to 0 when t goes to ∞ for all controls for which the ratio $\frac{A_t}{K_t}$ is strictly positive and strictly lower than a_{max}.

Denote by α the controls $(k, c^P, c, \beta, \beta_A)$. For $\varepsilon > 0$, let us define the set

$$\mathcal{A}_\varepsilon = \{\alpha \text{ such that } \varepsilon K_t^{(\alpha)} \leq \int \exp(\rho\omega) d\mathbb{P}_t^{(\alpha)}(\omega) \leq (a_{max} - \varepsilon) K_t^{(\alpha)} \text{ for } t \geq 0\}$$

and the set $\mathcal{A}$ as the union of $\mathcal{A}_\varepsilon$. We will prove that for every control $\alpha \in \mathcal{A}$, we have the transversality condition $\lim_{t \to +\infty} e^{-\rho t} \left(\frac{\log K_t^\alpha}{\rho} + v(a_t^\alpha) \right) = 0$ where

$$a_t^\alpha = \frac{\int \exp(\rho\omega) d\mathbb{P}_t^{(\alpha)}(\omega)}{K_t^{(\alpha)}}.$$

Take $\alpha \in \mathcal{A}$. There is $\varepsilon > 0$, such that $\alpha \in \mathcal{A}_\varepsilon$. Because v is continuous, it is bounded by a constant C_ε on the interval $[\varepsilon, a_{max} - \varepsilon]$, and we have

$$e^{-\rho t} v(a_t^\alpha) \leq e^{-\rho t} C_\varepsilon.$$

On the other hand, because the consumption processes are non-negative, the capital process $K^{(\alpha)}$ given in (20) is dominated by the Geometric Brownian motion

$$K \exp[(\mu - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A].$$

Consequently, we have

$$\mathbb{E}\left[e^{-\rho t} \log(K_t)\right] \leq \mathbb{E}\left[e^{-\rho t} (\log(K) + (\mu - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A)\right] \text{ which converges to 0.}$$

QED

8.2.3 Proof of Proposition 4 for power utility

. By analogy with the symmetric information case, we make the following guess

$$c_t^P = \gamma K_t, k_t = \frac{u^{-1}(\omega_t)}{a}, c_t = \frac{\rho}{y} k_t \text{ and } \beta_A = (1 - \alpha)\omega_t.$$

Since the IC constraint binds, the dynamics of ω_t are given by

$$d\omega_t = \rho(\omega_t - u(c_t))dt + \rho\sigma k_t u'(c_t)dZ_t + (1-\alpha)\omega_t\sigma_A dZ_t^A.$$

Inserting the above expressions of k_t and c_t we obtain:

$$\frac{d\omega_t}{\omega_t} = \rho\left(1 - \left(\frac{\rho}{ay}\right)^{1-\alpha}\right)dt + \sigma\varphi dZ_t + \sigma_A(1-\alpha)dZ_t^A,$$

where $\varphi = (1-\alpha)\left(\frac{\rho}{a}\right)^{1-\alpha} y^\alpha$. Since aggregate consumption is proportional to aggregate capital, K_t follows a geometric Brownian motion:

$$\frac{dK_t}{K_t} = gdt + \sigma_A dZ_t^A,$$

where $g = \mu - \gamma - \frac{\rho}{y}$. By integration we obtain

$$K_t = K\exp\left[(g - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A\right],$$

and we can compute the principal's value function:

$$V = \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t}\gamma^{1-\alpha}\frac{K_t^{1-\alpha}}{1-\alpha}\,dt\right].$$

We obtain

$$V = u(K)\frac{\rho\gamma^{1-\alpha}}{\rho - (1-\alpha)(g - \alpha\frac{\sigma_A^2}{2})}.$$

Since $g = \mu - \gamma - \frac{\rho}{y} \leq \mu$, our assumption $\rho > (1-\alpha)(\mu - \alpha\frac{\sigma_A^2}{2})$ implies that V is finite. We can then compute $A_t = \int u^{-1}(\omega_t)d\mathbb{P}_t(\omega_t)$.

$$A_t = A\exp\left[\left(\frac{\rho}{1-\alpha}\left(1 - \left(\frac{\rho}{ay}\right)^{1-\alpha}\right) + \frac{\sigma^2\varphi^2\alpha}{2(1-\alpha)^2} - \frac{(1-\alpha)}{2}\sigma_A^2\right)t + \sigma_A Z_t^A\right].$$

Since our conjecture implies that A_t is proportional to K_t, it must be that

$$g - \alpha\frac{\sigma_A^2}{2} = \frac{\rho}{1-\alpha}\left(1 - \left(\frac{\rho}{ay}\right)^{1-\alpha}\right) + \frac{\sigma^2}{2}\left(\frac{\rho}{ay}\right)^{2(1-\alpha)} y^2\alpha. \tag{64}$$

Therefore, $V(K,\mathbb{P}) = u(K)v(a)$ where

$$v(a) = \sup_{\gamma,y}\left(\frac{\rho\gamma^{1-\alpha}}{\rho - (1-\alpha)(g - \alpha\frac{\sigma_A^2}{2})}\right)$$

under constraint (64). This constraint is compatible with $\gamma > 0$ only if

$$\mu - \frac{\rho}{y} - \frac{\alpha\sigma_A^2}{2} - \frac{\rho}{1-\alpha}\left(1 - \left(\frac{\rho}{ay}\right)^{1-\alpha}\right) - \frac{\sigma^2\alpha}{2}y^2\left(\frac{\rho}{ay}\right)^{2(1-\alpha)} > 0.$$

Note that when α tends to 1, this condition coincides with the condition for the logarithmic case: $a < a_{max}$. We now prove that the function $V = u(K)v(a)$ is a smooth solution of the HJB equation that satisfies the condition of Proposition 3.

We first compute the derivatives of $V(K,\mathbb{P}) = u(K)v(a)$ where $a = \frac{A}{K}$.

$$V_K = K^{-\alpha}\left(v(a) - \frac{av'(a)}{1-\alpha}\right), V_{KK} = K^{-(1+\alpha)}\left(-\alpha v(a) + \frac{2\alpha}{1-\alpha}av'(a) + \frac{1}{1-\alpha}a^2v''(a)\right),$$

$$\partial_\omega \nabla V(\omega) = \frac{K^{-\alpha}}{1-\alpha}[(1-\alpha)\omega]^{\frac{\alpha}{1-\alpha}}v'(a), \partial^2_{\omega\omega}\nabla V(\omega) = \frac{K^{-\alpha}}{1-\alpha}[(1-\alpha)\omega]^{\frac{2\alpha-1}{1-\alpha}}\alpha v'(a),$$

$$\partial^2_{\omega K}\nabla V(\omega) = \frac{K^{-(1+\alpha)}}{1-\alpha}[(1-\alpha)\omega]^{\frac{\alpha}{1-\alpha}}(-\alpha v'(a) - av''(a)),$$

and

$$\partial^2_{\omega\omega'}\nabla^2 V(\omega,\omega') = \frac{K^{-(1+\alpha)}}{1-\alpha}[(1-\alpha)\omega]^{\frac{\alpha}{1-\alpha}}[(1-\alpha)\omega']^{\frac{\alpha}{1-\alpha}}v''(a).$$

Plugging these formulas in (27) gives the first order conditions, first with respect to c^P

$$\rho(c^P)^{-\alpha} = K^{-\alpha}\Big(v(a) - \frac{av'(a)}{1-\alpha}\Big),$$

which implies $c^P = \gamma(a)K$, then with respect to c:

$$-K^{-\alpha}\left(v(a) - \frac{av'(a)}{1-\alpha}\right) - \rho c^{-\alpha}\frac{K^{-\alpha}}{1-\alpha}[(1-\alpha)\omega]^{\frac{\alpha}{1-\alpha}}v'(a) - \sigma^2\rho^2k^2c^{-2\alpha-1}\frac{K^{-\alpha}}{1-\alpha}[(1-\alpha)\omega]^{\frac{2\alpha-1}{1-\alpha}}\alpha^2v'(a) = 0,$$

with respect to k:

$$\sigma^2\rho^2c^{-2\alpha}k\frac{K^{-\alpha}}{1-\alpha}[(1-\alpha)\omega]^{\frac{2\alpha-1}{1-\alpha}}\alpha v'(a) = \lambda,$$

and finally with respect to β_A :

$$\begin{aligned} 0 = &\frac{K^{-\alpha}}{1-\alpha}[(1-\alpha)\omega]^{\frac{\alpha}{1-\alpha}}(-\alpha v'(a) - av''(a)) + \beta_A\frac{K^{-\alpha}}{1-\alpha}[(1-\alpha)\omega]^{\frac{2\alpha-1}{1-\alpha}}\alpha v'(a) \\ &+ \frac{K^{-(1+\alpha)}}{1-\alpha}[(1-\alpha)\omega]^{\frac{\alpha}{1-\alpha}}\int \beta_A(\omega)[(1-\alpha)\omega]^{\frac{\alpha}{1-\alpha}}v''(a)d\mathbb{P}(\omega). \end{aligned}$$

It is easy to check that our conjectured solution

$$k(\omega) = \frac{u^{-1}(\omega)}{a}, c(\omega) = \frac{\rho k(\omega)}{y}, \beta_A = (1-\alpha)\omega$$

satisfies these first order conditions.

For the last step of the proof, we substitute the above controls into the HJB equation (27), and check that this equation is satisfied. We get

$$\rho\frac{K^{1-\alpha}}{1-\alpha}v(a) = \sup_{\gamma,y}\Big\{\rho\frac{K^{1-\alpha}}{1-\alpha}\gamma^{1-\alpha} + g\frac{K^{1-\alpha}}{1-\alpha}\big((1-\alpha)v(a) - av'(a)\big)$$
$$+\frac{\sigma_A^2}{2}\frac{K^{1-\alpha}}{1-\alpha}\big(a^2v''(a) + 2\alpha av'(a) - \alpha(1-\alpha)v(a)\big) + \frac{K^{1-\alpha}}{1-\alpha}\frac{\rho}{1-\alpha}\Big(1-\Big(\frac{\rho}{ay}\Big)^{1-\alpha}\Big)av'(a)$$
$$+\frac{K^{1-\alpha}}{1-\alpha}\Big(\frac{\sigma^2}{2}\Big(\frac{\rho}{ay}\Big)^{2(1-\alpha)}y^2 + \frac{\sigma_A^2}{2}\Big)a\alpha v'(a) + \frac{K^{1-\alpha}}{1-\alpha}\sigma_A^2\Big(-a\alpha v'(a) - a^2v''(a)\Big)$$
$$+\frac{K^{1-\alpha}}{1-\alpha}\frac{\sigma_A^2}{2}a^2v''(a)\Big\}$$

After simplifications, we obtain the following differential equation for v,

$$\rho v(a) = \sup_{\gamma,y}\Big\{\rho\gamma^{1-\alpha} + (1-\alpha)(g-\alpha\frac{\sigma_A^2}{2})v(a)$$
$$+av'(a)\Big(-g+\frac{\rho}{1-\alpha}\Big(1-\Big(\frac{\rho}{ay}\Big)^{1-\alpha}\Big) + \frac{\alpha}{2}\Big(\sigma^2\Big(\frac{\rho}{ay}\Big)^{2(1-\alpha)}y^2 + \sigma_A^2\Big)\Big)\Big\}.$$

QED

8.2.4 Proof of Proposition 9

Substituting γ from equation (43) into equation (31), we have that the expected growth rate is

$$g_{SB} = \mu - \rho + \frac{\rho}{y + \frac{\sigma^2}{\rho}y^3} - \frac{\rho}{y}.$$

After simplifications, this yields (45).

On the other hand, equation (38) implies $d\omega_t = d\log k(\omega_t)$. Using Ito's lemma and (31) and (32) yields (46).

QED

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