Dynamic contracting with many agents

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Abstract

We extend Merton (1969)'s analysis of capital allocation and consumption-savings choices to the case in which asset management is delegated to several privately informed agents. With a continuum of agents, mean-field control techniques yield a simple and intuitive solution: capital reallocation is linear in relative performance, and managers' fees are proportional to assets-under-management. We show that these properties do not obtain in the single agent case. We also show that continuation utilities are exposed to idiosyncratic risk and increasingly unequal between managers. Finally, investment is lower than under symmetric information because incentive constraints reduce risk-sharing.

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1 Introduction

This paper starts from the canonical problem analyzed by Merton (1969), in which a risk averse agent dynamically chooses i) how much to consume or save and ii) how to allocate savings across risky assets. We revisit this problem to account for the fact that, in practice, the management of the investor's portfolio of risky assets is delegated to asset managers. Such delegation generates incentive problems, to the extent that the investor (the principal) and the asset managers (the agents) have different preferences and information sets. Thus, our goal is to study how optimal investment-consumption choices and portfolio allocations are altered by incentive constraints. We focus on the case in which the investor delegates the management of the assets to several asset managers, each with special expertise, and correspondingly private information, about a given asset class.

Our focus on investment delegation to several asset managers is in line with stylized facts. The principal in our model can be thought of as a sovereign wealth fund, a family office, a pension fund, or a foundation. The principal allocates his/her portfolio across several different classes of assets, e.g., private equity, private debt, real estate, or infrastructure. This allocation across assets coincides with an allocation across asset managers, investing in assets about which they have special expertise. For example, at the end of 2023, the Norwegian sovereign wealth fund had 742 billion kroner (approximately 67 billion US dollars), managed externally by 103 organisations.[1] Similarly, in 2025, the New Zealand Superannuation Fund used more than 40 different external asset managers.[2] Rose (2016) notes that "nearly all sovereign wealth funds make significant use of external managers, particularly for alternative asset classes that are difficult to access or demand labor intensive investment strategies, such as private equity, venture capital, and hedge fund strategies." This quote suggests that delegation is motivated by the need to rely on the skills of the managers, skills which the investor does not master. In the Invesco Global Sovereign Asset Management Study, respondents stated that they often relied on external asset managers, and one of the respondents declared:[3] "When... we have been forced to operate the business ... that has historically not worked well, so we are moving away from that model and trying to leave all management to general partners". However, when delegation is motivated by the superior understanding of assets by agents, this generates an information asymmetry between the investor and the asset managers. This information asymmetry implies that the optimal allocation

[1] See https://www.nbim.no/en/the-fund/how-we-invest/external-mandates. The Norwegian Ministry of Finance decides the allocation of the portfolio across broadly defined asset classes, and then the sovereign fund decides how to allocate funds within these classes to different asset managers.

[2] See https://nzsuperfund.nz/how-we-invest/investment-managers/our-managers/

[3] See https://ioandc.com/sovereign-wealth-funds-tap-external-managers-for-private-markets-payday/

of assets across managers is subject to incentive compatibility constraints.

We model the interaction between an investor and N managers investing in different asset classes, such as private equity, private debt, real estate, or infrastructure. Each agent operates his/her own production technology, which can be interpreted as the technology for which the agent has the expertise which the principal lacks. The output generated by each agent is proportional to the amount of capital under his/her management. For simplicity, all agents are equally skilled, i.e., they have the same expected return per unit. They, however, are subject to different random shocks: each agent's return is hit by an agent-specific idiosyncratic shock, as well as a common aggregate shock. The aggregate shock and the allocation of capital across agents are publicly observable, and capital allocation is decided by the principal.

To model information asymmetry between the investor and the asset managers, we assume that agents privately observe their idiosyncratic shock, and correspondingly their return, which they can secretly divert and consume, as in Bolton and Scharfstein (1990).[4] We study the optimal mechanism designed by the investor (the principal) to allocate capital and compensation to the asset managers (the agents).[5] Applying the revelation principle, we study direct revelation mechanisms in which incentive compatibility constraints are imposed, so that agents truthfully report to the principal their privately observed returns. To analyze incentive constraints, we rely on the martingale techniques introduced in contract theory by Sannikov (2008). Incentive compatibility implies that agents' utilities must be sensitive to idiosyncratic shocks. Intuitively, after good performance agents are rewarded, while after bad performance they are punished. Differences in performance, therefore, generate endogenous heterogeneity among agents.

Solving the optimal control problem of the principal is difficult. To restore tractability, we consider the limit case of a continuum of agents, which enables us to rely on mean-field techniques.[6] The distribution of the continuation

[4]Another possible specification is that agents privately observe the effort they exert to improve returns, as in Holmström and Tirole (1997). In both specifications, the contract must be designed under the incentive constraint that agents "do the right thing", i.e., reveal privately observed performance or exert effort. While, we assume privately observed performance, privately observed effort (in line with Holmström and Tirole (1997)) would lead to similar results. DeMarzo and Fishman (2007a) and DeMarzo and Sannikov (2006) assume returns are privately observed by the agent, but show this leads to similar results as when effort is privately observed.

[5]The principal signs one contract with each agent, the collection of these contracts is the mechanism.

[6]In physics, mean-field theory studies the behavior of models where a large number of particles interact randomly by studying a simpler model where each particle interacts with an average distribution (a mean-field) of particles. The mathematical theory of mean-fields is presented in Lasry and Lions (2007), Cardaliaguet (2012), and Carmona and Delarue (2018). As discussed in more detail below, we take a mean-

utilities of the different agents is a state variable of the control problem of the principal. With a finite number of agents, this state variable is simply a vector. With a continuum of agents, the distribution of utilities across agents is an infinite dimensional object. Moreover, with aggregate risk it evolves stochastically. To analyze the optimal mechanism in that context we need to perform Itô calculus with infinite dimensional variables and to write a Hamilton Jacobi Bellman equation in which the value function is a functional. To do so we rely on on mean-field techniques. This enables us to obtain quasi analytical solutions yielding the following economic insights:

- Although on the equilibrium path performance only reflects luck, the incentive compatibility constraint implies that agents with better (resp. worse) idiosyncratic performance are rewarded (resp. punished). This generates exposure to idiosyncratic risk, reducing welfare relative to the first-best, in which idiosyncratic risk is fully mutualized, as in Borch (1962).

- A novel finding of the present paper is that the link between performance and rewards goes through capital allocation. Incentive compatibility implies that agents with good performance are allocated more capital than agents with bad performance.[7] This implies that capital allocation becomes more and more heterogeneous with time, with successful managers being allocated an increasingly larger fraction of capital than unsuccessful ones. We show that capital reallocation among managers is linear in their relative performance, and this enables us to write the dynamics of assets under management as a drifted Brownian motion with constant parameters.

- Also for incentives, managers' compensation is increasing in their capital under management. In line with stylized facts, in the optimal mechanism, asset managers' fees are proportional to their assets under management.

- Incentives do not only affect the allocation of capital among agents, they also lower overall investment in risky capital. The intuition is the following: Due to incentive constraints, the risk of capital investment is imperfectly shared among agents. Correspondingly, holding capital comes at the cost of greater risk exposure than in the first best allocation. The higher cost of holding capital implies that it is optimal to undertake less capital investment than in the first best allocation.

field *control* approach, which is the appropriate tool to study mechanism design and contracting, and is mathematically different from mean-field *games*, appropriate to study equilibrium.

[7] Allocating capital as a function of performance is optimal in spite of the fact that performance just reflects luck, and does not reflect any advantage in terms of skills and productivity.

- In the optimal mechanism , the sensitivity of agents' pay-offs to their idiosyncratic risk depends on their initial outside utility. When the initial outside utility of the agents is low, the principal can design the contract to extract a lot of rents from them. To do so, the principal makes the agents' utilities very sensitive to their idiosyncratic risk, which relaxes the incentive constraint. So, as the initial outside utility of the agents declines, the volatility of their compensation increases.

The simple and intuitive results we obtain with a continuum of agents are in stark contrast with those obtaining with a single agent.[8] With a single agent, the ratio of consumption to capital fluctuates randomly, and so does the sensitivity of the agent's utility to performance. In contrast, with a continuum of agents, consumption is a constant fraction of capital, and sensitivity to performance is constant. Intuitively, the reason for the difference between the one-agent case and the many-agents case is the following: With a single agent (or a small number of agents), the resources needed to provide individual incentives are significant relative to the total amount of resources available for investment and consumption. This generates a direct interaction between the individual incentive problem of an agent and the overall investment-consumption choice of the principal. In contrast, with a continuum of agents, agents' idiosyncratic shocks, and correspondingly, the resources needed to incentivize an individual agent, are negligible relative to aggregate resources. So, the individual incentive problems of the agents are separable from the investment-consumption problem of the principal. It is this separability that makes the problem tractable.

Our theoretical findings are consistent with stylized facts about asset management. Our model implies that asset managers with better performance should attract more capital, and that asset management fees should be proportional to assets under management, which is in line with the empirical results of Chevalier and Ellison (1997) and Sirri and Tufano (1998). Thus, in our model as in practice, money chases performance, not unlike momentum strategies. Yet, in our model all managers are equally skilled, so that realized performance does not reflect skills and does not predict future performance, which is consistent with the empirical finding that fund performance is hard to predict out of sample (see, e.g., Jones and Mo (2021)). Our theoretical model also implies that the distribution of capital and fees across managers should be skewed to the right, which is in line with empirical findings (see Bai et al, 2024). Collecting panel data on the allocation of large investors (e.g., Sovereign Wealth Funds) to different asset managers and asset classes, would

[8]The characterization of the dynamic optimal contract between a **risk averse** principal and a single agent is very difficult and has not been offered by previous literature (e.g., in Sannikov (2008) the principal is risk-neutral). But, of course, investor's risk aversion is a necessary ingredient of the Merton (1969) problem we revisit, in which risk-return tradeoffs play a key role.

enable one to conduct further empirical tests of our theory. For example, one could test the implications of our model for the joint dynamics of the cross-section of agents' performances and capital allocation.

Section 2 discusses the relationship between our paper and the literature. Section 3 presents the model. Section 4 considers the symmetric information benchmark, characterizing the first-best optimal allocations. Section 5 turns to the case of asymmetric information with a finite number of agents, characterizing, in particular, the incentive compatibility condition for a general number of agents, and the optimal mechanism with a single agent. Section 6 turns to the continuum of agents case, characterizing, in particular, the functional Hamilton Jacobi Bellman equation. Section 7 presents an explicit solution of the optimal control problem for logarithmic utilities. Section 8 presents extensions of our basic model. Section 9 concludes. Appendix A provides the proofs and propositions not provided in the main text. Appendix B details the numerical analysis of the one-agent case. Appendix C offers an introduction to the mean-field tools that underlie our analysis.

2 Literature

2.1 Continuous time finance and economics

Our analysis is at the cross of three strands of the continuous-time literature in finance and economics:

First, our analysis of dynamic portfolio allocations and consumption-savings choices is in the line of Merton (1969, 1973, 1987). The contribution of our analysis relative to these papers is to allow for asymmetric information between investors and asset managers, implying that capital allocation is not only driven by risk and return considerations but also by incentive considerations.

Second, to factor in these incentive constraints, we take an optimal dynamic contracting approach, in line with the seminal contributions of DeMarzo and Fishman (2007a, 2007b) and Sannikov (2008).[9] Two key differences between our paper and that literature are that we study the optimal contract between i) a *risk-averse* principal and ii) *several* agents. Considering a risk-averse principal is necessary to study a Merton problem in which the principal is an investor who evaluates risk-return tradeoffs. Considering

[9]See also DeMarzo and Sannikov(2006), Biais, Mariotti, Plantin, and Rochet (2007), Biais, Mariotti, Rochet, Villeneuve (2010), DeMarzo, Fishman, He and Wang (2012), Feng and Westerfield (2021), Yang (2020), Di Tella and Sannikov (2021), Dai, Wang, and Yang (2024), and Gryglewicz and Mayer (2023). In the present paper, as in Biais, Mariotti, Rochet, and Villeneuve (2010) and DeMarzo, Fishman, He, and Wang (2012), the scale of operations is determined by the optimal contract and is useful to provide incentives. He (2009) offers an interesting alternative approach in which firm size is affected by unobservable agent's effort. This differs from our model in which firm size is directly controlled by the principal, and what is unobservable is output.

several agents is appropriate to fit the stylized fact that the majority of large investors, such as sovereign wealth funds, family offices, or pension funds, rely on several asset managers. These assumptions lead to a challenging mathematical problem, but we are able to obtain tractability by taking a mean-field approach.

Third, our analysis is related to the mean-field literature, see, e.g., Lasry and Lions (2007), Cardaliaguet (2012), and Carmona and Delarue (2018). We study a mean-field Control problem, which is related to but different from Mean Field Games. Achdou et al. (2022) use mean-field games in a macroeconomic model to study market equilibrium. In contrast, we use mean-field control in a contract-theory model to study incentive-constrained optimal allocations. The methodological difference between mean-field games and mean-field control is illustrated by the difference between the analysis of Achdou et al (2022) and our analysis:

- In the mean-field approach of Achdou et al. (2022), each agent solves a Hamilton Jacobi Bellman equation, and the equilibrium distribution of wealth among agents is determined by a Fokker-Planck equation.
- In our mean-field control approach, it is the principal who solves a Hamilton Jacobi Bellman equation, and the distribution of agents' utilities is a state variable of that optimization problem.

Thus, the methodological contribution of our paper is to offer i) the first analysis of a dynamic contracting problem with a continuum of agents as a functional Hamilton Jacobi Bellman equation,[10] and ii) a simple and intuitive solution of this problem in the logarithmic utilities case.

2.2 Asset management

Our theoretical analysis differs from those of Berk and Green (2004) and Binsbergen, Brandt and Koijen (2008) because we focus on information asymmetry between the investor and the asset managers.

- In Berk and Green (2004) the investor and the asset managers have the same information and both conduct learning about the managers' skills by observing realized returns. Thus, in Berk and Green (2004) portfolio allocation is first-best efficient, unlike in our model. Moreover, Berk and Green (2004) consider linear utilities. This contrasts with

[10] To handle functionals defined over the Wasserstein space, we rely on the notion of Lions gradient, which adapts standard differential calculus to the geometry of the set of probability measures (Carmona and Delarue, (2018), p. 378-379). Elie, Mastrolia, and Possamaï (2019) also analyze a moral hazard problem with infinitely many agents via a functional HJB equation, but restrict attention to finite-horizon models with CARA preferences and a linear-quadratic structure for an explicit resolution. Our contribution is to provide a tractable solution for an infinite-horizon model.

our risk-aversion assumption, which, in conjunction with our information asymmetry assumption, enables us to characterize inefficiencies in portfolio allocation and the investment-consumption choice.

- In the main analysis of Binsbergen, Brandt and Koijen (2008) the investor and the asset manager have the same information, so any inefficiency in capital allocation and diversification is due to suboptimal contracts. Binsbergen, Brandt and Koijen (2008) then analyze the case in which the investor does not know the risk aversion of the asset managers, but they don't study the incentive-constrained optimal contract between the investor and the agents.

He and Xiong (2013) and Ou-Yang (2003) analyze optimal contracts. In their models, however, there is only one agent, and capital is allocated once and for all at the beginning of the contract. So, our analysis of dynamic reallocation of capital among agents brings a new contribution relative to He and Xiong (2013) and Ou-Yang (2003).

2.3 Optimal contracts with many agents

As in Holmström (1982), the principal in our paper interacts with several privately informed agents. But the focus of Holmström (1982) is very different from ours. In Holmström (1982) the actions of the agents interact in the determination of aggregate output. This can generate free-riding, which is a major issue in Holmström (1982). In contrast in our paper, agents' individual returns are independent from one another (conditional on the aggregate shock), so there is no free-riding, which simplifies the analysis. On the other hand, our paper studies the dynamic allocation of capital across agents, an issue that is absent from Holmström (1982).

3 Model

We extend Merton(1969)'s analysis of the consumption-investment and portfolio allocation choices to a principal-agent context: the investor (the principal) delegates the management of the portfolio of assets to several asset managers (the agents).

3.1 Preferences and Technology

Time is continuous: $t \in (0, \infty)$. The agents indexed by $i = 1, ...N$ and the principal are infinitely lived with discount rate ρ and utility $\rho u(c)$, where $u(.)$ is increasing and concave.[11] To illustrate our results, we will take the specification $u(c) = \log c$. However, our general characterization results are valid for more general utility functions.

[11] Multiplying the current utility by ρ is a useful convention that generates simpler formulas, For example, the intertemporal utility of a constant consumption flow c is just $\int_0^\infty \rho e^{-\rho t} u(c) dt = u(c)$.

Each of the N agents has mass $1/N$, so that the total mass of the agents is equal to 1. We will later take the limit when N goes to infinity to consider a continuum of agents. There is a single good which can be consumed or invested in N stochastic constant returns to scale technologies.[12] The principal does not have the expertise to operate the production technologies, but the agents do. More precisely, each agent has the expertise to operate one of the N technologies. If agent i invests an amount $\frac{k_t^i}{N}$ in his/her own technology.[13] This generates instantaneous output:

$$dY_t^i = \frac{k_t^i}{N}[\mu dt + \sigma dZ_t^i + \sigma_A dZ_t^A], \tag{1}$$

where μ is the expected rate of return of the technology and $(Z_t^A, Z_t^i)_{i=1,...N}$, are independent Brownian motions. (dZ_t^A) is an aggregate productivity shock, to which all projects are equally exposed, while $(dZ_t^i)_{i=1,...N}$ are idiosyncratic, project-specific, productivity shocks. $(\mathcal{F}_t)_{t\geq 0}$ denotes the augmented filtration generated by the $N+1$-dimensional Brownian motion $(Z_t^A, Z_t^1, \ldots, Z_t^N)_{t\geq 0}$. All processes introduced in this section are assumed to be square-integrable and progressively measurable with respect to $(\mathcal{F}_t)$.

3.2 Aggregate capital and output

The total amount of capital in the economy at time t is

$$K_t := \frac{1}{N}\sum_{i=1}^{N} k_t^i. \tag{2}$$

If, at time t, the consumption flow of agent i is $\frac{c_t^i}{N}$ and that of the principal is c_t^P, the law of motion of aggregate capital is

$$dK_t = \sum_{i=1}^{N} dY_t^i - \left(\frac{1}{N}\sum_{i=1}^{N} c_t^i + c_t^P\right) dt. \tag{3}$$

Summing over the individual outputs given in (1) and using (2), the aggregate output is given by

$$dY_t = \sum_i dY_t^i = K_t(\mu dt + \sigma_A dZ_t^A) + \frac{\sigma}{N}\sum_i k_t^i dZ_t^i. \tag{4}$$

[12] With constant returns to scale, assuming that the principal is risk-averse is important for obtaining interior solutions. If the principal had linear utility, the problem would be degenerate, with either 0 investment or infinite investment.

[13] We denote by $\frac{k_t^i}{N}$ the capital invested by agent i. The intuition is that $\frac{1}{N}$ is the weight of the agent. This formulation facilitates the analysis of the limit case of a continuum of agents, in which $N \to \infty$, so that there is an infinite number of agents, each with negligible mass.

Substituting (4) into (3), the dynamics of aggregate capital rewrites

$$dK_t = \left(\mu K_t - \frac{1}{N}\sum_{i=1}^{N} c_t^i - c_t^P\right) dt + \sigma_A K_t dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^{N} k_t^i dZ_t^i. \tag{5}$$

Equation (5) is a resource constraint that states that net aggregate investment is equal to total output minus total consumption.[14]

3.3 Pareto frontier

A consumption path for an agent is a progressively measurable nonnegative process $(c_t)_t$ such that

$$\mathbb{E}\int_0^\infty \rho e^{-\rho s}|u(c_s)|ds < +\infty.$$

For a given consumption path (c_t^i), the expected continuation utility of agent i at time t is

$$\omega_t^i := \mathbb{E}_t \int_t^\infty \rho e^{-\rho(s-t)} u(c_s^i)ds. \tag{6}$$

We seek to characterize the Pareto frontier of the economy by computing the value function V of the principal, defined as the maximum expected utility she can obtain with an initial amount of capital K when agent $i \in (1, ..., N)$ gets the initial expected utility ω^i. Let us denote by $\mathbb{W}$, the vector of initial expected utilities $(\omega^1, ..., \omega^N)$. $\mathbb{W}$ are the outside utilities that the agents could obtain initially if they refused to contract with the principal. They can be interpreted as reflecting what the agents could obtain from contracting with other principals competing with the principal analyzed in our model.

The value function of the principal is obtained by finding capital and consumption paths (k_t^i, c_t^i, c_t^P) that maximize the expected utility of the principal.

$$\mathbb{E}\int_0^\infty \rho e^{-\rho t} u(c_t^P)dt, \tag{7}$$

subject to the relevant constraints. With symmetric information, these constraints are the capital allocation constraint (2), the resource constraint (5), and the initial conditions: $K_0 = K$ and $\omega_0^i = \omega^i$, for $i = 1, ..., N$. Under information asymmetry, there will also be incentive compatibility constraints.

We assume that the agents and the principal commit to the contract signed on the date 0.[15] We first consider the case of symmetric information, in which

[14]For brevity, we do not explicitly include depreciation. μ can be understood as capital productivity net of depreciation.

[15]Hence, agents cannot walk away from the contract after time 0 to take an outside option. While this assumption may be unrealistic, it is needed to simplify the analysis, which is already quite complex. Note however that our commitment assumption does not necessarily imply that agents' continuation value becomes arbitrarily negative. As explained below, if capital productivity is high enough all agents' utilities go to infinity.

idiosyncratic shocks and thus individual outputs are publicly observable. This case will then serve as a reference point for the case in which agents privately observe their idiosyncratic shocks.

3.4 Promise keeping

Denote by M_t^i the expectation of the lifetime utility of agent i conditional on information at time t :

$$M_t^i \equiv \mathbb{E}_t \left[\int_0^\infty \rho e^{-\rho s} u(c_s^i) ds \right] = \int_0^t \rho e^{-\rho s} u(c_s^i) ds + e^{-\rho t} \omega_t^i.$$

Consequently, the dynamics of ω_t^i is

$$d\omega_t^i = \rho \left[\omega_t^i - u(c_t^i) \right] dt + e^{\rho t} dM_t^i.$$

Since M_t^i is the conditional expectation of an integrable random variable, it is a martingale. Therefore, by the Martingale Representation Theorem, dM_t^i can be written as a linear combination of Brownian shocks at time t, $(dZ_t^j)_{j=1,\dots,N}$ and (dZ_t^A). Thus, the dynamics of i's continuation utility can be written as

$$d\omega_t^i = \rho \left[\omega_t^i - u(c_t^i) \right] dt + \frac{\sigma}{N} \sum_j \beta_t^{ij} dZ_t^j + \sigma_A \beta_t^{A,i} dZ_t^A, \tag{8}$$

where β_t^{ij} and $\beta_t^{A,i}$ are adapted to $(\mathcal{F}_t)$. $\beta_t^{A,i}$ can be interpreted as the share of the volatility of the aggregate shock that is borne by agent i, while β_t^{ij} is the exposure of agent i to the idiosyncratic shock of agent j. They are controls chosen by the principal. In line with the contract theory literature, hereafter we refer to equation (8) as the "promise keeping condition ".

4 Optimal allocations under symmetric information

The focus of this paper is on the asymmetric information case. However, the symmetric information case offers a useful benchmark in which our methodological approach can be explained in simple terms, paving the way to the analysis of the more complex asymmetric information case.

4.1 The control problem of the principal

The state variables of the principal's control problem are the capital stock K_t and the vector $\mathbb{W}_t \equiv (\omega_t^1, ..., \omega_t^N)$ of promised utilities. The controls k_t^i, c_t^i, c_t^P, β_t^{ij}, and $\beta_t^{A,i}$ only depend on K_t and $\mathbb{W}_t$. The value function of the principal is

$$V(K, \mathbb{W}) = \sup_{k,\beta,\beta^A,c,c^P} \mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} u(c_t^P) dt \right], \tag{9}$$

subject to the capital allocation constraint (2), the resource constraint (5), the promise keeping condition (8), and the initial conditions: $K_0 = K$ and

$\mathbb{W}_0 = \mathbb{W}$. The associated Hamilton-Jacobi-Bellman equation is

$$\rho V(K, \mathbb{W}) = \sup_{k,\beta,\beta^A,c,c^P} \rho u(c^P) + \sum_{i=1}^{N} \left[V_{\omega^i} \rho \left(\omega^i - u(c^i) \right) \right] + V_K \left(\mu K - c^P - \sum_{i=1}^{N} \frac{c^i}{N} \right)$$
$$+ \sum_{i=1}^{N} \left[V_{\omega^i K} \left(\beta^{A,i} \sigma_A^2 K + \sigma^2 \sum_j \frac{k^j \beta^{ij}}{N^2} \right) \right] + \frac{1}{2} V_{KK} \left[\sigma_A^2 K^2 + \sigma^2 \sum_i \left(\frac{k^i}{N} \right)^2 \right]$$
$$+ \frac{1}{2} \sum_{i,j} V_{\omega^i \omega^j} \left(\sigma_A^2 \beta^{A,i} \beta^{A,j} + \sigma^2 \sum_\ell \frac{\beta^{i\ell} \beta^{j\ell}}{N^2} \right) \tag{10}$$

where the sup is subject to the capital allocation constraint (2). The shape of the Hamilton-Jacobi-Bellman equation (10) implies that in the first-best, consumption allocation and capital allocation are separable, since the consumption allocation controls c^i and c^P appear only in the first line of (10), while the capital allocation controls k^i only appear in the second line of (10).

The verification theorem in stochastic control (see, e.g., Yong and Zhou (1999) chapter 4) yields our first proposition:

Proposition 1 *If there exists a twice continuously differentiable solution V of the HJB equation (10) that satisfies the transversality condition,*

$$\lim_{t \to \infty} e^{-\rho t} \mathbb{E}[V(K_t, \mathbb{W}_t)] = 0,$$

it is the value function of the principal's problem (9).

We now show that, when u is logarithmic, the solution is explicit.

4.2 An explicit solution when utility is logarithmic

Merton (1969) studied a simpler form of our model in which the investor did not need agents and directly managed the portfolio of assets. In that case, when $u(c) = \log c$, the solution always exists and the optimal allocation is such that:

- capital is equally allocated across assets,[16]
- the consumption flow is proportional to the capital stock,
- the capital stock follows a geometric Brownian motion,
- the value function of the principal satisfies $V(K) = \log mK$ for some constant m.

[16]This is because, in our model, all assets have the same expected return and volatility.

We now show that a similar result obtains when the principal must delegate asset management to N agents under symmetric information. However, a solution to this problem exists only if the capital stock is large enough to provide the continuation utilities promised to the agents. More specifically, we need

$$K_t > \frac{A_t}{m_N}$$

for all $t \geq 0$, where A_t is the cross sectional average of the inverse utilities of the agents:

$$A_t := \frac{1}{N} \sum_i \exp(\omega_t^i)$$

and the constant m_N is given by

$$m_N = \rho \exp(\frac{1}{\rho}(\mu - \rho - \frac{1}{2}(\sigma_A^2 + \frac{\sigma^2}{N}))). \tag{11}$$

The next proposition characterizes the first-best allocations between the principal and the agents:

Proposition 2 *When u is logarithmic, The optimal mechanism exists if and only if*

$$A < m_N K. \tag{12}$$

In that case the value function of the principal is

$$V(K, \mathbb{W}) = \log \left(m_N K - \frac{1}{N} \sum_i \exp \omega^i \right) = \log K + \log(m_N - \frac{A}{K}), \tag{13}$$

and the optimal mechanism has the following features for all t:

- *The inverse utility of each agent is proportional to aggregate capital:*

$$\exp \omega_t^i = \frac{K_t}{K_0} \exp \omega_0^i, \tag{14}$$

  The average inverse utility of the agents, A_t, is also proportional to aggregate capital:

$$A_t = \frac{K_t}{K_0} A_0. \tag{15}$$

- *Capital is equally allocated across agents*

$$k_t^i \equiv K_t, i = 1, ..., N. \tag{16}$$

- *The consumption flows of the principal and the agents are proportional to the aggregate capital stock.*

$$c_t^P = \gamma K_t, c_t^i = \gamma^i K_t, i = 1, ..., N. \tag{17}$$

- *Aggregate capital and individual capital follow a geometric Brownian motion:*

$$\frac{dK_t}{K_t} = \frac{dk_t^i}{k_t^i} = \left(\mu - \gamma^P - \frac{1}{N}\sum_{i=1}^{N}\gamma^i\right)dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_{i=1}^{N} dZ_t^i. \quad (18)$$

- *The aggregate consumption rate is equal to the discount rate*

$$\gamma^P + \frac{1}{N}\sum_{i=1}^{N}\gamma^i = \rho \quad (19)$$

- *The sensitivity of agent i's continuation utility to the Brownian shocks is given by*

$$\beta^{A,i} \equiv \beta^{ij} \equiv 1, \quad i,j = 1,...,N. \quad (20)$$

Condition (12) states that the amount of initial capital is sufficiently large relative to the agents' reservation utilities, making it possible to design a mechanism that offers the agents their reservation utility while leaving positive consumption for the principal. Equation (19) states that the aggregate consumption rate is equal to the discount rate, as in the original Merton model. Equation (15) states that the ratio of A_t to aggregate capital is constant and Equation (13) shows that the value function depends on the vector of continuation utilities $\mathbb{W}_t$ only through this ratio. Similar properties will obtain in the asymmetric information case, which will greatly enhance the tractability of the problem. Proposition 2 illustrates that with log utility and symmetric information, the capital allocation problem and the consumption allocation problem can be solved separately.

- The sole purpose of capital allocation is to maximize diversification. Thus, as stated in Equation (16), capital is allocated equally between projects run by the different agents.

- As can be seen in (16) and (17), an agent's consumption and capital only depend on aggregate capital. Hence, an agent's individual performance affects him/her only via its impact on aggregate investment and therefore capital. Thus, as stated in Equation (3), idiosyncratic risks and aggregate risk are equally shared among all participants. There is full mutualization. However, that the individual performance of an agent equally affects this agent and the other agents is at odds with stylized facts. As shown in the next section, with asymmetric information we will obtain the more plausible result that an agent's performance affects this agent more than it affects the others.

The proof of Proposition 2 is in the appendix. We adopt the guess-and-verify method that will be used throughout the paper. First, we compute the value function associated with the guessed optimal controls, given in (16), (17), and (18), and show that this yields the guessed value function, given in (13). Second, we verify that this value function satisfies the Bellman equation (10) and then invoke Proposition 1.

Under symmetric information, there is no qualitative difference between the model with a finite number of agents and the model with a continuum of agents. As explained below (in the next section and in Appendix C.1), when N goes to infinity the empirical distribution of agents' continuation utilitoes, $\mathbb{W}$, converges to a continuous distribution $\mathbb{P}$. Relying on this, to obtain the optimal allocation with a continuum of agents, it suffices to take the limits when N goes to infinity in equations (13), (18), and (19). With a continuum of agents, equation (18) simplifies because

$$\frac{\sigma}{N}\sum_{i=1}^{N} dZ_i \to 0,$$

since the Brownian motions corresponding to the idiosyncratic shocks of the agents are independent and square-integrable. The mean-field limit analyzed in Appendix C implies that first best allocations with a continuum of agents are characterized by the following proposition:

Proposition 3 *When u is logarithmic and there is a continuum of agents, under symmetric information the optimal mechanism exists if and only if*

$$A = \int exp\omega d\mathbb{P}(\omega) < m_\infty K,$$

where $m_\infty = \rho \exp \frac{1}{\rho}(\mu - \rho - \frac{\sigma_A^2}{2})$. The value function of the principal is

$$V(K, \mathbb{P}) = \log(m_\infty K - A).$$

The optimal mechanism has the following features for all t:

- *The inverse utility of each agent is proportional to aggregate capital:*

$$\exp \omega_t = \frac{K_t}{K_0} \exp \omega_0,$$

- *Capital is equally allocated across agents*

$$k_t \equiv K_t.$$

- *The consumption flow of the principal is proportional to the aggregate capital stock.*

$$c_t^P = \gamma K_t.$$

- *The consumption flows of the agents are proportional to their inverse utility*

$$c_t = (\rho - \gamma)\frac{\exp \omega_t}{\int \exp \omega d\mathbb{P}_t(\omega)}.$$

- *Aggregate capital and individual capital follow a geometric Brownian motion:*

$$\frac{dK_t}{K_t} = \frac{dk_t}{k_t} = (\mu - \rho)\, dt + \sigma_A dZ_t^A.$$

- *Aggregate risk is equally shared, while idiosyncratic shocks are fully insured*

$$\beta^A(\omega) \equiv 1, \beta(\omega) \equiv 0.$$

Proposition (3) is literally the limit of Proposition (2) when $N \to \infty$.[17] As we shall see in the following, under asymmetric information, we are no longer able to solve the optimal allocation with a finite number of agents. We will have to directly solve the continuum of agents' limit of the Hamilton Jacobi Bellman equation.

Moreover, under symmetric information, each agent's inverse utility is proportional to aggregate capital, while under asymmetric information, as shown below, incentive compatibility constraints prevent inverse utilities from being proportional across agents. Nevertheless, the proportionality between capital and inverse utility will be maintained at the **individual** level. Capital allocation will therefore be an important tool for providing incentives to the agents.

5 Optimal allocations under asymmetric information

We now turn to the case in which agents privately observe their idiosyncratic shock (dZ_t^i), while aggregate shocks and individual capital are still publicly observable.[18] Private observation of individual output generates incentive problems because we assume agents can secretly divert and consume some of

[17]As long as individual shocks are fully diversified, the allocation of capital to each agent is irrelevant in the limit problem. The allocation given in the proposition is the limit of the capital allocation on the finite agent problem, which is uniform. Another difference between the finite number of agents' case and the continuum of agents' case is that in the former $\beta = 1$, while in the latter $\beta = 0$. With a finite number of agents, individual idiosyncratic shocks are relatively large, so it is optimal to share them equally, implying $\beta = 1$. With a continuum of agents, each idiosyncratic shock is negligible relative to the aggregate, so there is nothing to be gained by sharing it. On the other hand, there is something to be lost by exposing an individual agent to his/her individual shock.

[18]While our analysis assumes agents privately observe their output after it is realized and then can secretly divert it, similar results would obtain in a moral hazard setting à la Holmström and Tirole (1997). In that specification, agents would privately observe if they exert costly effort at time t, and effort would improve the instantaneous drift of their output process. The incentive compatibility condition that the agent exerts the right level of effort leads to a similar incentive compatibility condition to that we obtain. DeMarzo and Sannikov (2006) explicitly demonstrate the similarity between the two forms of information asymmetry in the context of a one-agent model with risk neutrality.

the output they generate (as in Bolton and Scharfstein (1990) and DeMarzo and Sannikov (2006).)

By the revelation principle, it is without loss of generality to focus on truthful revelation mechanisms. Thus, the second best allocation is achieved by the mechanism optimally mapping the history of aggregate and idiosyncratic shocks into allocations, under the resource constraint and the incentive compatibility constraint that agents truthfully report their shocks.

Heuristically, the sequence of events at time t is the following:

- Each agent i operates the technology, with capital k_t^i.
- Shocks dZ_t^i and dZ_t^A realize. Each agent i reports $d\hat{Z}_t^i$ to the planner, transferring the corresponding output $d\hat{Y}_t^i$.
- As a function of the history of shocks, the planner allocates to the agent i, his/her period consumption flow c_t^i, capital for the next period k_{t+dt}^i, and promised continuation utility ω_{t+dt}^i.

5.1 Incentive compatibility and the Hamilton Jacobi Bellman equation

By the promise keeping condition (8), when truthfully revealing dZ_t^i and anticipating that other agents report truthfully, agent i gets

$$\rho u(c_t^i)dt + \frac{\sigma}{N}\sum_{j=1}^{N}\beta_t^{ij}dZ_t^j + \sigma_A\beta_t^A dZ_t^A. \tag{21}$$

Similarly, when under-reporting: $d\hat{Z}_t^i = dZ_t^i - \delta dt$, agent i gets

$$\rho u(c_t^i + \sigma\delta\frac{k_t^i}{N})dt + \frac{\sigma}{N}\beta_t^{ii}(dZ_t^i - \delta dt) + \frac{\sigma}{N}\sum_{j\neq i}\beta_t^{ij}dZ_t^j + \sigma_A\beta_t^A dZ_t^A. \tag{22}$$

Incentive compatibility requires (21) to be larger than (22). Because u is concave, this inequality holds for all $\delta \geq 0$ under the condition stated in the following proposition:

Proposition 4 *The incentive compatibility condition holds if and only if*

$$\beta_t^{ii} \geq \rho k_t^i u'(c_t^i). \tag{23}$$

Since $k_t^i u'(c_t^i) > 0$, (23) implies that the dynamics of ω_t^i in (8) must be affected by the idiosyncratic shock of agent i. Substituting the first best allocation obtained for logarithmic utility, given in (16), (17), and (3), into (23), implies $\gamma^i > \rho$. This, however, is inconsistent with (19) and $\gamma^P > 0$. So, at least for logarithmic utility functions, the first best allocation is not incentive compatible. This reflects that in the first best $\beta_t^{ii} = 1$, i.e., the idiosyncratic risk of each agent is fully mutualized by all the agents. This contradicts the incentive compatibility constraint (23), which requires that agents be sufficiently exposed to their own shock.

Moreover, the incentive compatibility condition (23) gives rise to a trade-off between risk sharing and investment. Exposing risk-averse agents to idiosyncratic shocks reduces their utility. To reduce that exposure, one needs to reduce β_t^{ii}. However, due to (23), reducing β_t^{ii} requires increasing c_t^i and/or reducing k_t^i. As shown below, the second best will thus differ from the first best by tilting the consumption trade-off towards consumption, leading to less investment than in the first best.

With asymmetric information, the Hamilton Jacobi Bellman equation is still given by (10) but the maximization problem is subject to the incentive compatibility condition (23), in addition to the capital allocation constraint (2). Condition (23) implies that, in contrast to the first best, capital allocation and consumption allocation decisions are no longer separable. This is because, while in (10) the terms involving capital k^i and consumption c^i are added to one another, in (23) capital k^i and consumption c^i multiply one another. Non separability of capital and consumption decisions makes the optimal control problem much less tractable. To illustrate this difficulty, we now study the simplest version of this problem: with a single agent ($N = 1$) and logarithmic utilities. It turns out that this is a difficult problem, whose solution can only be obtained numerically.

5.2 The control problem of the principal with a single agent

When $N = 1$, the Hamilton Jacobi Bellman equation (10) becomes:

$$\rho V(K, \omega) = \sup_{\beta, \beta^A, c, c^P, k} \rho u(c^P) + \rho V_\omega [\omega - u(c)] + V_K \left(\mu K - c^P - c\right)$$

$$+V_{\omega K} \left(\beta^A \sigma_A^2 + \beta \sigma^2\right) K + \frac{1}{2} V_{KK} \left[\sigma^2 + \sigma_A^2\right] K^2 + \frac{1}{2} V_{\omega\omega} \left(\sigma_A^2 (\beta^A)^2 + \sigma^2 \beta^2\right), \tag{24}$$

where the sup is taken under the capital allocation constraint

$$k = K,$$

and the incentive compatibility constraint

$$\beta \geq \rho k u'(c). \tag{25}$$

With logarithmic utilities, optimal mechanisms are homogeneous: if the capital stock is multiplied by some number λ, the optimal consumption flows are also multiplied by λ, and the utilities of both the agent and the principal are increased by an amount $\log \lambda$. That is

$$V(\lambda K, \omega + \log \lambda) = V(K, \omega) + \log \lambda.$$

Taking $\lambda = 1/K$, this yields

$$V(1, \omega - \log K) = V(K, \omega) - \log K. \tag{26}$$

Denoting

$$a \equiv \omega - \log K \tag{27}$$

and $v(a) \equiv V(1, a)$, (26) rewrites

$$V(K, \omega) = \log K + v(a) \tag{28}$$

Homogeneity can be exploited by redefining control variables as:

$$\gamma = \frac{c^P}{K}, y = \frac{\rho K}{c}, \tag{29}$$

where y is the right hand side of the incentive compatibility condition (25). Moreover, formulas (27) and (28) imply that the partial derivatives of V are:

$$V_\omega = v'(a), V_{\omega\omega} = v''(a), V_{\omega K} = -v''(a)/K \tag{30}$$

and

$$V_K = \frac{1 - v'(a)}{K}, V_{KK} = \frac{v''(a) + v'(a) - 1}{K^2}. \tag{31}$$

Using these formulas, the HJB equation becomes after simplifications:

$$\rho v(a) = \sup_{\gamma, y \leq \beta, \beta^A} \rho \log \gamma + \rho v'(a)(a - \log \frac{\rho}{y}) + [1 - v'(a)][\mu - \gamma - \frac{\rho}{y}]$$

$$+[v''(a) + v'(a) - 1]\frac{\sigma^2 + \sigma_A^2}{2} + \frac{v''(a)}{2}[(\sigma_A^2((\beta^A)^2 - 2\beta^A) + \sigma^2(\beta^2 - 2\beta)].$$

As we show in the appendix, it is optimal to take $\beta_A = 1$ and $\beta = y$, which leads to our next proposition, whose proof is in the appendix:

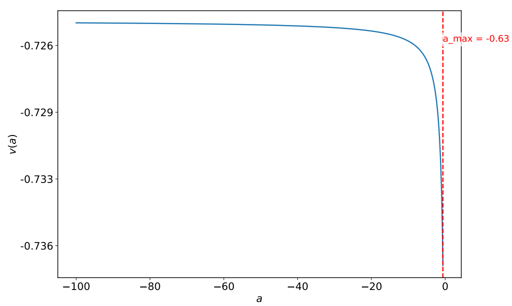
Proposition 5 *Consider the case where there is only one agent. When the principal and agent have logarithmic utility, the aggregate risk is equally shared between the principal and the agent ($\beta^A \equiv 1$) and the incentive compatibility condition is always binding ($\beta \equiv y$). The principal's value function solves the following Hamilton Jacobi Bellman equation:*

$$\rho v(a) = \sup_{\gamma, y} \rho \log \gamma + \rho v'(a)(a - \log \frac{\rho}{y})$$

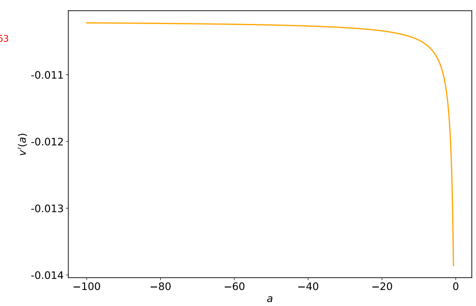
$$+(1 - v'(a))(\mu - \gamma - \frac{\rho}{y} - \frac{\sigma^2 + \sigma_A^2}{2}) + \frac{\sigma^2}{2} v''(a)(1 - y)^2. \tag{32}$$

As stated in the proposition, aggregate risk is equally shared by the principal and the agent as in the first best (i.e., $\beta^A = 1$). This is because aggregate risk is publicly observable. So there is nothing to be gained by deviating from the first-best risk aggregate risk-sharing policy.

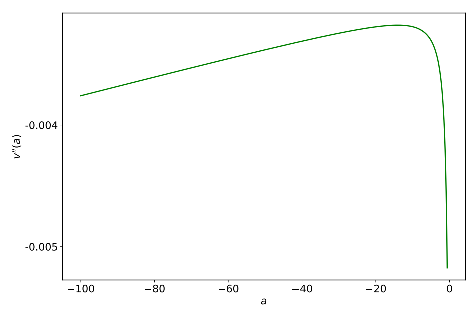
In spite of the simplification induced by homogeneity, equation (32) remains a complex second-order differential equation. It can only be solved numerically. Appendix B explains how this numerical solution is obtained. In the following Figure 1, based on the numerical analysis, we illustrate how the value function of the principal depends on state variable a.



(a) $v(a)$



(b) $v'(a)$



(c) $v''(a)$

Figure 1: Illustrative Plot of the value function $v(a)$ and its first and second derivatives for the parameters $K_0 = 100$, $\rho = 0.04$, $\mu = 0.15$, $\sigma = 0.1$, $\sigma_A = 0.05$ implying that $a_{\max} = 0.6251$.

Taking the first order conditions in the Hamilton Jacobi Bellman equation (32), yields the optimal controls as a function of the state variable a_t: $(\gamma_t \equiv \gamma(a_t), y_t \equiv y(a_t))$.

$$\frac{\rho}{\gamma_t} = 1 - v'(a_t), \tag{33}$$

and

$$\frac{\rho v'(a_t)}{y_t} + \frac{\rho(1 - v'(a_t))}{y_t^2} + \sigma^2 v''(a_t)(y_t - 1) = 0. \tag{34}$$

Moreover, the dynamics of the state variable are the following:

$$da_t = [\rho(a_t - \log\frac{\rho}{y_t}) - (\mu - \gamma_t - \frac{\rho}{y_t} - \frac{\sigma^2 + \sigma_A^2}{2}]dt + \sigma(y_t - 1)dZ_t, \tag{35}$$

with initial condition $a_0 = a$. Together, equations (33) to (35) yield the dynamics of the state variable and the controls in the optimal contract for one agent. Again, we are not able to solve analytically for these dynamics, but the analysis implies that a_t and therefore γ_t and y_t fluctuate randomly over time.

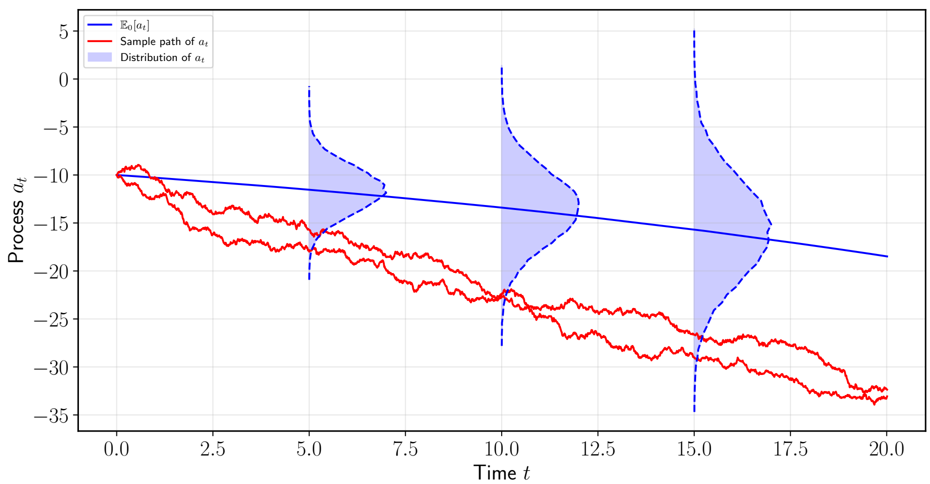


Figure 2: Dynamics of the state variable a_t for $\rho = 0.04$, $\mu = 0.15$, $\sigma = 0.1$, $\sigma_A = 0.05$ and $a_0 = -10$. The distributions are generated by 50000 samples. We use a root-finding algorithm to obtain y_t.

This is illustrated in Figure 2, which plots two randomly drawn sample paths for a_t as well as the distribution of a_t after 5 periods, 10 periods, and 15 periods (estimated from 50000 simulated sample paths.) The figure illustrates that a_t is subject to large fluctuations. Correspondingly, with a single agent, in the optimal mechanism consumption and exposure to shocks undergo large random changes through time. As shown below, this contrasts with the continuum of agents' case, in which, for logarithmic utilities, the state variable $a_t \equiv a$ and the optimal controls (γ_t, y_t) are constant over time. This leads to an intuitive and quasi-explicit characterization of the optimal mechanism.

6 The case of a continuum of agents

We now consider the limit case of a continuum of agents, i.e., $N \to \infty$. While, as shown below, this restores tractability, this also makes the problem mathematically challenging because it amounts to a control problem in an infinite dimensional space. This section starts by sketching the mean-field limit of the problem, then presents the functional form of the Hamilton Jacobi Bellman equation in this context. We tried to keep the main text simple and intuitive. In Appendix C we offer an introduction to mean-field theory and explain how we use it to study the continuum of agents limit of our model.

6.1 The mean-field limit

With a finite number of exchangeable agents, the value function is a symmetric function of the vector of utilities $\mathbb{W}$. It can be equivalently expressed as a function of capital stock K and the empirical distribution of the agents' continuation utilities. Due to the presence of the common noise Z^A, we expect that the empirical distribution weakly converges to a random distribution $\mathbb{P}_t$ when N goes to infinity. Thus, the state variables at time t are K_t and $\mathbb{P}_t$, where the latter is the conditional distribution of the continuation utility of the agents given the filtration generated by the aggregate shocks. As explained in Appendix C1, using results from the propagation of chaos theory (Snitzman (1991)), the aggregate consumption of agents at date t converges to

$$\int c(K_t, \mathbb{P}_t; \omega) d\mathbb{P}_t(\omega),$$

while the resource constraint (5) converges to

$$dK_t = \left(\mu K_t - \int c(K_t, \mathbb{P}_t; \omega) d\mathbb{P}_t(\omega) - c^P(K_t, \mathbb{P}_t)\right) dt + \sigma_A K_t dZ_t^A, \quad (36)$$

which reflects that, when the number of agents goes to infinity, idiosyncratic noise diversifies, so that the only stochastic term in the dynamics of aggregate capital is the aggregate shock dZ_t^A. Similarly, the dynamics of an agent's continuation utility only contains two stochastic terms, the aggregate shock dZ_t^A and the agent's idiosyncratic shock dZ_t:[19]

$$d\omega_t = \rho\left(\omega_t - u(c_t)\right) dt + \sigma\beta dZ_t + \sigma_A \beta_A dZ_t^A. \quad (37)$$

The mechanism maps i) aggregate capital, ii) the distribution $\mathbb{P}$, and iii) the continuation utility of agent ω, into capital and consumption allocations, as well as agents' exposures to shocks:

$$X \equiv (c^P(K, \mathbb{P}), k(K, \mathbb{P}, \omega), c(K, \mathbb{P}, \omega), \beta_A(K, \mathbb{P}, \omega), \beta(K, \mathbb{P}, \omega)).$$

So, the principal value function V depends on the aggregate capital K and the distribution of continuation utilities $\mathbb{P}$, and the control problem of the principal thus writes

$$V(K, \mathbb{P}) = \sup_X \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c_t^P) dt\right],$$

where the expectation is taken over aggregate shocks, the state equations are given by the resource constraint (36) and the promise keeping condition (37), and maximization is taken over the set of controls that satisfy for all t the incentive compatibility condition (23) and the capital allocation constraint

$$K_t = \int_{\mathbb{R}} k(K_t, \mathbb{P}_t; \omega)\, d\mathbb{P}_t(\omega). \quad (38)$$

[19]In line with Holmström's (1979) informativeness principle, it would be suboptimal to make ω_t^i depend on the idiosyncratic shocks of other agents, about which the agent does not have any private information.

6.2 The functional Hamilton Jacobi Bellman equation

Since the principal value function V depends on the distribution of continuation utilities $\mathbb{P}$, to write the Hamilton-Jacobi-Bellman equation, we need to apply Itô's formula to V. This requires differential calculus in the space $\mathcal{P}_2$ of square integrable measures on $\mathbb{R}$, and in particular the first and second order derivatives of V with respect to the distribution $\mathbb{P}$. The expressions of these derivatives involve the gradient ∇V and the Hessian $\nabla^2 V$ of functional V, which are defined precisely in Appendix C.2. This allows extending the classical notions of first and second derivatives to functionals defined in the space of probability measures. Relying on these tools, we can write the Hamilton Jacobi Bellman equation in the continuum of agents' case:

$$\rho V(K,\mathbb{P}) = \sup_{c^P,c,\beta^A,\beta} \left\{ \rho u(c^P) + V_K \left(\mu K - c^P - \int c(\omega) d\mathbb{P}(\omega) \right) \right.$$

$$+\frac{1}{2} V_{KK} \sigma_A^2 K^2 + \int \partial_\omega \nabla V(\omega) \rho(\omega - u(c(\omega))) d\mathbb{P}(\omega)$$

$$+\frac{1}{2} \int \partial^2_{\omega\omega} \nabla V(\omega) \left(\sigma^2 \beta^2(\omega) + \sigma_A^2 \beta_A^2(\omega) \right) d\mathbb{P}(\omega) + \int \partial^2_{\omega K} \nabla V(\omega) \sigma_A^2 \beta_A(\omega) K d\mathbb{P}(\omega)$$

$$\left. +\frac{1}{2} \int \int \partial^2_{\omega\omega'} \nabla^2 V(\omega,\omega') \sigma_A^2 \beta_A(\omega) \beta_A(\omega') d\mathbb{P}(\omega) d\mathbb{P}(\omega') \right\}, \qquad (39)$$

where the supremum is subject to the incentive compatibility condition and the capital allocation constraint.

When $\partial^2_{\omega\omega} \nabla V(\omega, K) < 0$ (which will be checked ex post), it is optimal to bind the incentive compatibility condition (23). This determines the control $\beta(\omega)$:

$$\beta(\omega) = \rho k(\omega) u'(c(\omega)) > 0. \qquad (40)$$

Because of risk aversion, constrained optimality implies that the sensitivity of each agent's continuation utility to its current performance is exactly what is needed for incentive compatibility.

A difficulty is that the capital allocation constraint (38) mixes control variables and state variables. To deal with this difficulty, we introduce a related, unconstrained, problem as follows: for each function λ (defined on the product space $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$), which we call the *Lagrange multiplier*, we consider the control problem

$$V_\lambda = \sup_X \int_0^\infty e^{-\rho t} \left(\log c^P(K_t, \mathbb{P}_t) + \lambda(K_t, \mathbb{P}_t) \left(K_t - \int k(., \omega) d\mathbb{P}_t(\omega) \right) \right) dt. \qquad (41)$$

In the appendix, in Propositions 16 and 17 we show that the solution V of the principal's optimization problem can be obtained via V_λ.

7 An explicit solution when utility is logarithmic

7.1 Guess

From now on, we assume that $u(c) = \log c$. By analogy with the first best, we guess (and then verify) that the value function of the principal only depends on two scalars, aggregate capital K_t and an exhaustive statistics for $\mathbb{P}_t$, namely the average inverse utility of agents:

$$A_t = \int \exp \omega d\mathbb{P}_t(\omega). \tag{42}$$

As in the case of one agent (see equation (28)), we guess (and check ex post) that the value function can be written as

$$V(K_t, \mathbb{P}_t) = \log K_t + v(a_t), \tag{43}$$

where

$$a_t = \log \frac{A_t}{K_t}. \tag{44}$$

This notation is in line with the case $N = 1$, where we had set $a_t = \omega_t - \log K_t$ in equation (27).

7.2 Optimal policy under the guess

Under our guess, the partial derivatives of V with respect to K have the same expression as in the case $N = 1$, given in (31). Using our guess, we can also compute the gradient of the value function and its derivatives. First, note that the gradient of V with respect to $\mathbb{P}$ is proportional to $\exp \omega$:

$$\nabla V(\omega) = \frac{v'(a) \exp \omega}{A}. \tag{45}$$

This implies

$$\partial_\omega \nabla V = \partial^2_{\omega\omega} \nabla V = \nabla V, \tag{46}$$

and

$$\partial_{\omega K} \nabla V = -\frac{v''(a) \exp \omega}{KA}. \tag{47}$$

Finally,

$$\nabla^2 V(\omega, \omega') = \frac{v''(a) \exp \omega \exp \omega'}{A^2} - \frac{v'(a) \exp \omega \exp \omega'}{A^2} = \partial^2_{\omega\omega'} \nabla^2 V(\omega, \omega'). \tag{48}$$

Like in the case of one agent, it is convenient to redefine the control variables as follows:

$$\gamma = \frac{c^P}{K}, y = \frac{\rho k}{c}.$$

Substituting formulas (45) to (48) into the Hamilton Jacobi Bellman equation (39), and using the binding incentive constraint (40) and the first order conditions, we obtain the optimal policy under our guess:

Proposition 6 *Under our guess, the optimal mechanism has the following properties: The consumption of the principal is proportional to aggregate capital, with a proportionality constant that depends on the state variables only via the ratio* $a \equiv \log \frac{A}{K}$*:*

$$c^P(K, \mathbb{P}) = \gamma(a)K = \frac{\rho}{1 - v'(a)} K. \tag{49}$$

Similarly, the consumption of an agent with continuation utility ω *is proportional to the capital allocated to that agent, with a proportionality factor that depends only on the state variables via* a*:*

$$c(\omega, K, \mathbb{P}) = \frac{\rho}{y(a)} k(\omega, K, \mathbb{P}). \tag{50}$$

The capital allocated to an agent with continuation utility ω *is proportional to* $\exp \omega$*:*

$$k(\omega, K, \mathbb{P}) = \exp(\omega - a), \tag{51}$$

Finally, the optimal exposure of agents to their idiosyncratic shock only depends on a *and equals*

$$\beta \equiv y(a),$$

while the exposure to aggregate risk is the same for all agents, i.e.,

$$\beta_A \equiv 1.$$

The optimal exposure to aggregate risk is the same in the second-best as in the first-best. This is because the aggregate risk is publicly observable. Therefore, it can be shared optimally without jeopardizing incentives. Equation (50) states that the consumption of the agents' is proportional to their own capital, while equation (49) states that the consumption of the principal is proportional to aggregate capital, and in both cases, the proportionality factor depends on the state variables (aggregate capital and distribution of utilities) only via the ratio a.

Proposition6 also states that $\beta \equiv y(a)$. That is the exposure of agents to their idiosyncratic risk is constant relative to ω and thus equal across agents. Moreover, it also depends on the state variables only via the ratio a. In the next subsection, we show that in fact a is **constant** along the optimal trajectory.

7.3 Solution of the Hamilton Jacobi Bellman equation under the guess

Substituting optimal consumption, capital allocation, and sensitivity to aggregate shocks, along with (45) - (48), into the Hamilton Jacobi Bellman equation, the latter simplifies to:

$$v(a) = \sup_{\gamma,y}\Big(\log\gamma + \frac{\mu-\gamma-\frac{\rho}{y}-\frac{\sigma_A^2}{2}}{\rho} + v'(a)(a - \log\frac{\rho}{y} - \frac{\mu-\gamma-\frac{\rho}{y}-\frac{\sigma_A^2+\sigma^2y^2}{2}}{\rho})\Big). \tag{52}$$

The next proposition, proved in the appendix, establishes that the solution of problem (52) is such that the ratio $a_t = \log\frac{A_t}{K_t}$ is constant along optimal trajectories.

Proposition 7 *With logarithmic utilities, in the optimal mechanism, the ratio of average continuation utility to aggregate capital, $a_t = \log\frac{A_t}{K_t} \equiv a$, is constant along optimal trajectories.*

This property significantly simplifies the problem and enables us to characterize the optimal mechanism in quasi closed form, as shown below. Because of this simplification, the problem with a continuum of agents is much more tractable than the problem with a finite number of agents. The optimal controls are also constant along optimal trajectories, and the Hamilton Jacobi Bellman equation (52) is much simpler than its counterpart with one agent (32). With a continuum of agents, the Hamilton Jacobi Bellman equation is of order 1, while it is of order 2 with one agent as can be seen in equation (32). In addition, the next proposition establishes that the Hamilton Jacobi Bellman equation for a continuum of agents, (52), is equivalent to a static optimisation problem:

Proposition 8 *The Hamilton Jacobi Bellman equation (52) is equivalent to a static optimization problem:*

$$v(a) = \sup_{\gamma,y}\Big(\log\gamma + \frac{1}{\rho}(\mu - \frac{\sigma_A^2}{2} - \gamma - \frac{\rho}{y})\Big), \tag{53}$$

under

$$a = \log\frac{\rho}{y} + \frac{1}{\rho}(\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2+\sigma^2y^2}{2}). \tag{54}$$

Moreover, the function v is a decreasing concave function.

The interpretation of Proposition 8 is natural. The objective function (53) of the static problem is equal to the utility of the principal when γ and y are constant over time and $K = 1$: log consumption plus the ratio of the expected growth rate minus a risk premium over the discount rate. Similarly, the constraint (54) expresses the average utility of agents when γ and y are constant over time and $K = 1$. Compared to the principal's utility, in the agent's utility the propensity to consume is $\frac{\rho}{y}$ instead of γ and the risk premium is $\frac{\sigma_A^2+\sigma^2y^2}{2}$ instead of $\frac{\sigma_A^2}{2}$. The difference in terms of risk stems from the fact that the principal is only exposed to the aggregate risk (as the impact of idiosyncratic risk on output diversifies away), while each agent is also exposed to his/her own idiosyncratic risk for incentive reasons. The relation between the static optimisation problem (53) and the HJB equation (52) becomes clear: the right hand side of (52) can be interpreted as the Lagrangian of (53) by nothing that the Lagrange multiplier equals $v'(a)$.

Using Proposition 8, the next proposition states the condition under which the above optimization problem admits a solution.

Proposition 9 *The static optimisation problem stated in Proposition 8 admits a solution if and only if*

$$a < a_{max} \equiv \sup_y [\log \frac{\rho}{y} + \frac{1}{\rho}(\mu - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2})] \tag{55}$$

Intuitively, there exists a solution to the optimization problem if the amount of capital available K is large enough that the principal can deliver the distribution of continuation utilities $\mathbb{P}$ (with $a = \log \frac{\int \exp \omega d\mathbb{P}(\omega)}{K}$), while retaining a positive consumption, i.e. $\gamma > 0$. Therefore a must be less than the value a_{max} obtained by setting $\gamma = 0$ on the right-hand side of (54).

7.4 Verification

The above analysis implies that, with logarithmic utilities, our initial guess (43), leads to the value function given in Proposition 8, which satisfies the Hamilton Jacobi Bellman equation (84). To guaranty that this function defines the principal's value, it remains to check the transversality condition. This is the subject of the following proposition, whose proof is identical to its symmetric information counterpart and, therefore, omitted.

Proposition 10 *The value function $V(K, \mathbb{P}) = \log K + v(a)$, where*

$$a = \log \frac{\int \exp \omega d\mathbb{P}}{K},$$

satisfies the transversality condition. Therefore, it is the solution of the control problem of the principal.

The function v is not explicit, except when $\sigma = 0$ in which case

$$v(a) = \log(m_\infty - \exp a),$$

corresponding to the first best value function characterized in Proposition 3, provided that (55) holds.

7.5 Consumption and exposure to idiosyncratic risk

The first order conditions of the static optimisation problem (53) give, after simple rearrangements

$$\gamma = \frac{\rho}{1 - v'(a)},$$

and

$$\frac{\sigma^2}{\rho}y^3 + y = 1 - \frac{1}{v'(a)}.$$

We can eliminate $v'(a)$ between these two conditions, implying an increasing relation between γ and y:

$$\gamma = \rho(1 - \frac{1}{\frac{\sigma^2}{\rho}y^3 + y}). \tag{56}$$

Since v is concave ($v'' < 0$), we see that both γ and y are decreasing in a. This yields our next proposition:

Proposition 11 *With logarithmic utility, both the consumption rate γ of the principal and the exposure of agents to their idiosyncratic risk, y, decrease with a.*

The interpretation of the proposition is the following. As the initial utility of agents increases, a increases. So, the larger a, the larger the utility agents can demand. To provide this utility, the principal must reduce his consumption and increase that of the agents, which reduces the agents' risk exposure. Proposition 11 is illustrated in Figure 3, which plots y as a function of a.[20]. Figure 3 is generated for the same parameter values as 4. The figure shows how, in the optimal mechanism, y decreases as a increases.

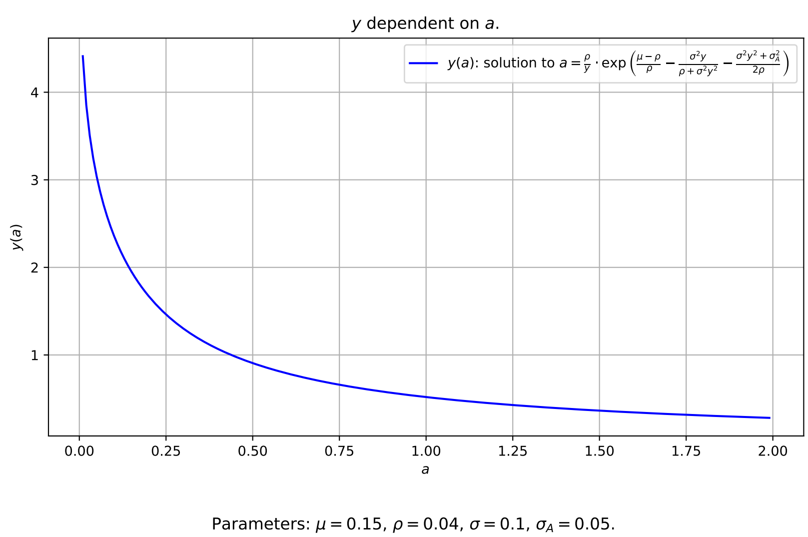


Figure 3: Idiosyncratic risk exposure as a function of bargaining power.

[20]As can be seen in equation (54), when a goes to 0, y must go to infinity

7.6 Dynamics of capital in the optimal mechanism

Based on the above analysis, the next proposition characterizes the dynamics of capital in the optimal mechanism.

Proposition 12 *With logarithmic utilities, in the optimal mechanism, the dynamics of aggregate capital is*

$$\frac{dK_t}{K_t} = g_{SB}dt + \sigma_A dZ_t^A, \tag{57}$$

where the expected growth rate g_{SB} is given by

$$g_{SB} = \mu - \rho - \frac{\rho\sigma^2 y}{\rho + \sigma^2 y^2}, \tag{58}$$

and the dynamics of individual capital is

$$\frac{dk_t}{k_t} = \frac{dK_t}{K_t} + \sigma y dZ_t. \tag{59}$$

In the first best, capital allocation was solely driven by diversification motives: This implied that capital was equally split across agents. In contrast, with asymmetric information, condition (59) shows that capital allocation is driven by incentives motives: Thus, the capital growth rate for agents with unexpectedly good (resp. bad) performance is larger (resp. lower) than the aggregate capital growth rate. This can be interpreted as a capital allocation rule that rewards performance relative to the market benchmark, and punishes under-performance relative to that benchmark.

Equation (58) implies that when $\sigma = 0$, so that there is no incentive problem, the growth rate is equal to its first best level $(\mu - \rho)$. However, when $\sigma > 0$, the expected growth is lower in the second best than in the first best. This is driven by the fact that incentive compatibility precludes perfect risk-sharing. To relax the incentive constraint, and thus improve risk-sharing, it is optimal to lower capital and raise consumption relative to the first best. This, in turn, reduces growth.

Equation (58) also implies that the aggregate expected growth rate g_{SB} is decreasing for $y < \sqrt{\rho}/\sigma$, and increasing for $y \geq \sqrt{\rho}/\sigma$. This is illustrated in Figure 4. To generate the figure, the expected productivity of the risky production technology, μ, is set at 15%, while the standard deviation of the idiosyncratic risk, σ, is set at 10%. Moreover, the agent discount rate, ρ, is set to 4%, while the standard deviation of the aggregate shock, σ_A, is set to 5%.

Figure 4 shows that the expected growth is non-monotonic in y. This is because an increase in y has two effects that go in opposite directions: On the one hand, other things equal an increase in y reduces agents' consumption, which increases growth. On the other hand, an increase in y increases the consumption of the principal, which reduces growth. As can be seen in the figure, for large values of y, the first effect dominates the latter.

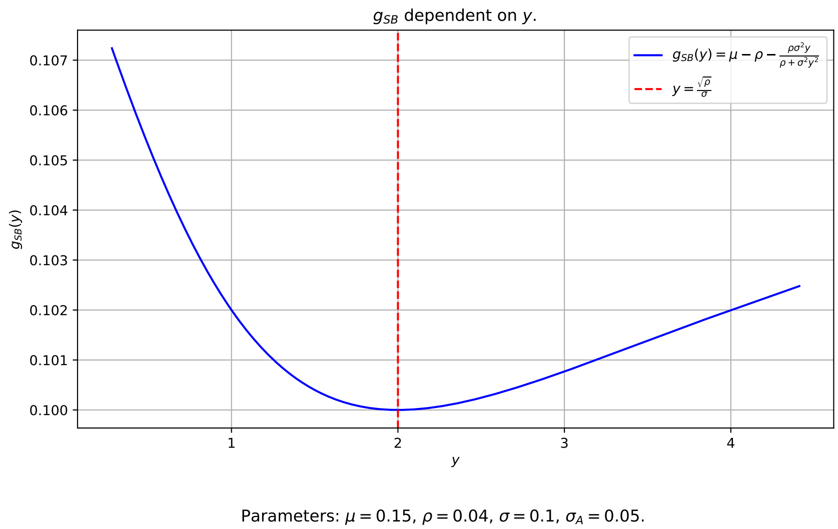


Figure 4: Expected growth as a function of idiosyncratic risk exposure.

7.7 Dynamics of utility in the optimal mechanism

The above analysis also yields the dynamics of continuation utility in the optimal mechanism, which is explained in the next proposition.

Proposition 13 *With logarithmic utility, in the second best agents' continuation utilities follow the drifted Brownian motion:*

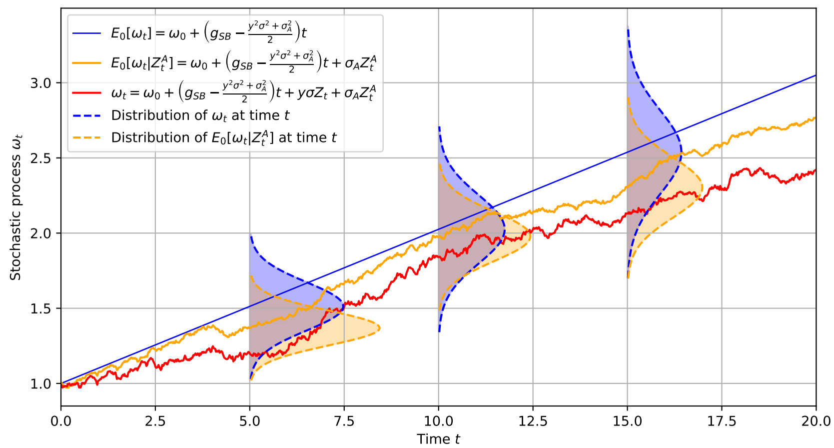
$$\omega_t = \omega_0 + \left(g_{SB} - \frac{y^2\sigma^2 + \sigma_A^2}{2} \right) t + y\sigma Z_t + \sigma_A Z_t^A. \tag{60}$$

Equation (60), illustrated in Figure 5, characterizes the dynamics of the continuation utility of the agents. Figure 5 is generated for the same parameter values as Figure 3 and Figure 4, except that in Figure 3 and Figure 4, a varied between 0 and 2 while in Figure 5, a is constant and set to 1.

The blue line in Figure 5 is the expectation of the time t continuation utility of an agent conditional on the information of time 0. This expectation is affine in t and equal to

$$\omega_0 + (g_{SB} - \frac{y^2\sigma^2 + \sigma_A^2}{2})t.$$

The slope reflects the opposite effects of expected aggregate growth, which tends to increase the utility of agents, and of costly risk bearing, which tends to reduce the utility of agents. If capital productivity μ is large enough, the positive effect of growth dominates the negative effect of costly risk-bearing, so that, in expectation, agent's utility tends to increase. This is the case for the parameters used to generate the figure. So, in the figure, the blue line is increasing.



Path and distribution of ω_t for $\omega_0 = 1$. Parameters: $\mu = 0.15$, $\rho = 0.04$, $a = 1$, $\sigma = 0.1$, $\sigma_A = 0.05$.

Figure 5: Dynamics of agents' continuation utility.

The blue bell shape in the figure is the distribution of agents' time t continuation utility around its expectation conditional on time 0 information. The dispersion around the expectation reflects the weighted sum of the two Brownian terms

$$y\sigma Z_t + \sigma_A Z_t^A.$$

The standard deviation of this Brownian term increases with (the square root of) time. Consequently, in Figure 5, the width of the blue bell shape increases with time, i.e., the inequality between agents increases with time. This increase in inequality is due to the incentive compatibility constraint, which implies that agents' continuation utilities must react to their idiosyncratic performance.

Similarly to the first best, if the capital productivity (μ) is high enough, so that the drift in ω_t is positive, all agents' utilities go to infinity.[21] The difference between the first and second best is that, in the latter, the productivity needed for the continuation utilities to go to infinity is greater, to compensate for the negative effect of exposure to the idiosyncratic risk.

The stochastic process in orange in the figure is the expected continuation utility of the agent at time t, conditional on one sample path for the realization of the aggregate shock Z_t^A, that is,

$$\omega_0 + (g_{SB} - \frac{y^2\sigma^2 + \sigma_A^2}{2})t + \sigma_A Z_t^A.$$

[21]Intuitively, this is due to the fact that the expectation of the continuation utility increases linearly with time, while the standard deviation increases as the square root of time. When t goes to infinity, the linear increase dominates the square root increase.

The orange bell shape is the distribution of the continuation utility at time t around this conditional expectation. The dispersion around this conditional expectation reflects only the idiosyncratic risk, multiplied by the exposure of the agent to this risk, that is,

$$y\sigma Z_t.$$

The blue bell shape shows more dispersion than the orange bell shape, since the former reflects uncertainty about Z_A^t and Z_t, while the latter only reflects uncertainty about Z_t. However, similar to the blue bell shape, the width of the orange bell shape increases with time, reflecting the increasing cross-sectional inequality between agents.

Finally, the stochastic process in red is the realization of the stochastic process of an agent's continuation utility (ω_t), conditional on one sample path for aggregate risk (Z_A^t), and one sample path for the agent's idiosyncratic risk (Z_t). Thus, Figure 5 illustrates that there is a simple relation between the continuation utility of an agent at date t and its performance over $(0, t)$. The total productivity of the agent on $(0, t)$ is $\mu t + \sigma Z_t + \sigma^A Z_t^A$. When there is no aggregate risk, optimal compensation implies a simple, affine relation between continuation utility ω_t and this performance measure, similarly to Holmström and Milgrom (1987). With aggregate risk, optimal compensation is more complex as it must also take into account aggregate performance.

Our analysis of the distribution of continuation utilities, combined with our result that $k(\omega) = \exp(\omega - a)$, where a is a constant, implies that the distribution of capital across agents is lognormal. Moreover, its variance across agents increases with time.

7.8 Information constrained Pareto frontier

The above analysis yields a characterization of the information-constrained Pareto frontier in the space of inverse utility. Substituting the expression of γ from (**??**) into the continuation value of the agent (60), and the value function of the principal (53), we obtain

$$\exp\omega = \frac{\rho}{y} k(\omega) \exp\left[\frac{g_{SB}}{\rho} - \frac{\sigma^2 y^2 + \sigma_A^2}{2\rho}\right], \tag{61}$$

where g_{SB} is the growth rate given in (58)) and

$$\exp V(K, \mathbb{P}) = \rho\left(1 - \frac{1}{y + \frac{\sigma^2}{\rho} y^3}\right) K \exp\left[\frac{g_{SB} - \frac{\sigma_A^2}{2}}{\rho}\right]. \tag{62}$$

Figure 6 depicts the Pareto frontier generated by equations (61) and (62). In general, the Pareto frontier depends on the utilities of all agents. In order to draw a figure in two dimensions, we consider the case in which all agents start with one unit of capital, and thus all have the same initial utility. Thus, the horizontal axis is the representative agent's lifetime discounted expected utility, while the vertical axis is the principal's lifetime discounted expected

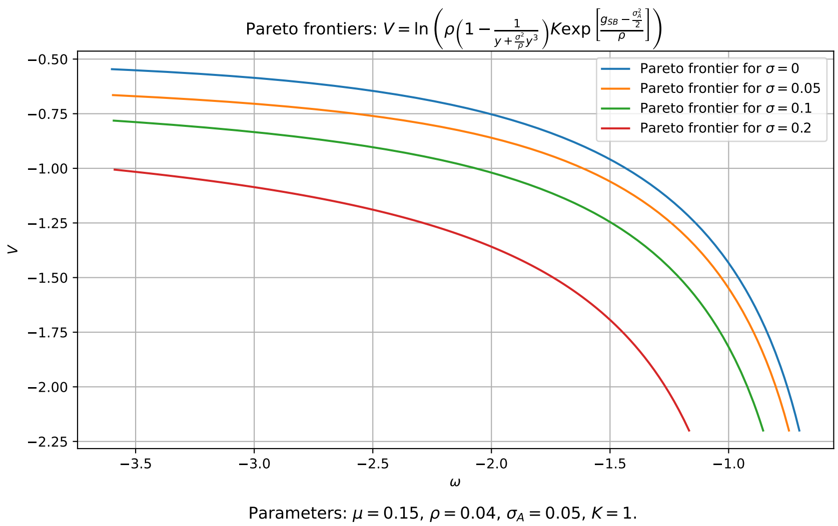


Figure 6: Pareto frontiers.

utility. Different points on the Pareto frontier correspond to different values of y. As y increases, we move along the Pareto frontier to the northwest, corresponding to an allocation that is more favorable to the principal and less favorable to the agent. The figure also depicts the Pareto frontier for different values of σ. This shows that as the idiosyncratic risk increases, making the moral hazard problem more severe, the Pareto frontier shifts downwards.

8 Extensions

This section discusses how our findings are affected by changes in assumptions regarding preferences and investment opportunities.

8.1 Risk neutral principal

First, we compare our above analysis, in which the principal is risk-averse, to the case in which the principal's utility is linear. The assumption that the principal is risk neutral does not fit well the portfolio management context we consider, yet it is useful to consider to understand the role of the risk aversion of the principal in our model. Analyzing our model with a risk neutral principal, we obtain the following proposition:

Proposition 14 *If the principal is risk neutral and $\mu > \rho + \frac{\sigma^2 + \sigma_A^2}{2}$, then the principal's value function is infinite.*

To put Proposition 14 into perspective, Table 1 presents the value function obtained in a dynamic investment context in the different cases obtained when i) the principal is risk neutral or risk averse, ii) capital and cash flow per period are constant (no growth) or can grow (growth), and iii) information is

Table 1: Principal value function depending on whether the principal is risk averse or not and whether asset size is fixed or can grow.

Principal utility	Linear	Logarithmic
Fixed asset size	$\sigma = 0$: Value of annuity $V = \frac{\mu K_0}{\rho}$ $\sigma > 0$, asymmetric information Sannikov (2008), V finite	$\sigma = 0$: Value of annuity $V = \frac{\log \mu K_0}{\rho}$
Asset growth (investment)	$\sigma = 0$: Gordon growth model If $\mu > \rho$, V infinite $\sigma > 0$, asymmetric information: If $\mu > \rho + \frac{\sigma^2 + \sigma_A^2}{2}$, V infinite (see Proposition 14)	$\sigma = 0$: Merton model V finite (see Proposition 2) $\sigma > 0$, asymmetric information: V finite (see Proposition 8)

symmetric or asymmetric. For simplicity, in the table (but not in the proof of the proposition), we set the outside utility of the agents (ω^i) to 0.

When $\sigma = 0$, there is no incentive problem and we have the first best. When there is no growth, the value function is the present value from the utility of consuming a constant flow. If the principal has linear utility, this is $V = \frac{\mu K_0}{\rho}$, while if the principal has logarithmic utility this is $V = \frac{\log \mu K_0}{\rho}$. With growth, when the principal has linear utility this is the Gordon growth model. In that model, if the growth rate is larger than the discount rate, then the value of the principal is infinite. In contrast, when the principal has logarithmic utility, we are in the Merton case, spelled out in Proposition 2, and the principal's value is finite. So we see that, without growth there is no major difference between the case in which the principal is risk neutral and the case in which the principal is risk averse. In contrast, when there is growth (and the growth rate is above the discount rate) the value of the principal is finite for logarithmic utility but infinite for risk neutrality. This is because, with linear utility, the principal finds it optimal to postpone consumption to accumulate capital and thus obtain arbitrarily large output and hence utility. In contrast, with logarithmic utility, it is not optimal to fully postpone consumption, since at 0 consumption the marginal utility from consumption is infinite. So, as in Merton (1969), the principal prefers to have a smooth consumption profile, consuming a fraction ρ of the output flow and investing the rest. This yields a finite value for the principal.

When $\sigma > 0$ and information is asymmetric, the comparison between a risk neutral and a risk averse principal is similar to its $\sigma = 0$ counterpart. Without growth, the value function of the principal is finite, irrespective of whether the principal is risk neutral (as in Sannikov (2008)) or risk averse. With growth (and if the growth rate is large enough), there is a striking contrast between the case in which the principal is risk neutral and the case in which the principal is risk averse: In the former (as stated in Proposition 14) the value of the principal is infinite, in the latter it is finite (as in Proposition 8). As in the symmetric information case, this is because a risk neutral principal obtains arbitrarily large utility by differing consumption and accumulating capital for a sufficiently long time.

The overall conclusion of this discussion is that, in a growth and investment context, the assumption that the principal is risk averse plays a key role. It is that assumption which leads to a smooth consumption profile and a finite value function for the principal.[22]

8.2 Risk-free asset

The second extension of our model is to the case in which, in addition to the risky investment opportunity, there is a risk-free asset. This is in line with the original Merton problem, in which there is a risk-free bond with deterministic return r. In this context, there is an additional instrument: the fraction $x \leq 1$ of total wealth K that is invested in risky technologies. To simplify the analysis, we consider the case without aggregate risk, i.e., we assume $\sigma_A = 0$. To analyze this problem, similarly to the case with no risk-free asset, we take a guess and verify approach. We guess that the optimal controls are given by

$$c_t = \frac{\rho}{y} k_t, \tag{63}$$

as in (50), and

$$k_t = x \exp(\omega_t - a),$$

where

$$a = \log(\int \exp \omega_t d\mathbb{P}_t) - \log K_t,$$

is the reduced state variable. Here, k_t is proportional to the fraction x of total wealth that is invested in the risky asset. As in the analysis above, y affects both the consumption of the agents (as implied by (63)) and the dynamics of their capital, which we guess to be

$$\frac{dk_t}{k_t} = (\mu x + r(1-x) - \frac{\rho x}{y})dt + \sigma y dZ_t. \tag{64}$$

[22]In Biais et al (2010), the principal is risk-neutral and there is investment and growth, yet the principal's value remains finite. This is because in that paper there is an additional constraint that the investment rate (and correspondingly the growth rate) must be lower than a constant which is itself lower than the discount rate of the principal.

The drift term of (64) is equal to the overall expected return on assets (which is the average weighted by x of the expected returns on the risky and risk-free assets) minus consumption. Finally, in line with the above analysis, we guess that the controls x and y are constant over time. Using the same method as in Section 7.3, the reduced form of the HJB equation becomes:

$$\rho v(a) = \sup_{x \leq 1, y, \gamma} \rho \log \gamma + (1 - v'(a))[\mu x + r(1-x) - \gamma - \frac{\rho x}{y}]$$
$$+ v'(a)(\rho a - \rho \log \frac{\rho x}{y} + \frac{\sigma^2 y^2}{2}).$$

The solution of this equation that satisfies the transversality condition is also the solution of a static optimisation problem

$$v(a) = \sup_{x \leq 1, y, \gamma} \log \gamma + \frac{1}{\rho}[\mu x + r(1-x) - \gamma - \frac{\rho x}{y}]$$

under the constraint

$$a = \log \frac{\rho x}{y} + \frac{1}{\rho}[\mu x + r(1-x) - \gamma - \frac{\rho x}{y} - \frac{\sigma^2 y^2}{2}].$$

The differences from the previous HJB equation (52) are very mild:

- The expected return on assets is $\mu x + r(1-x)$ instead of μ,
- the ratio of agents' consumption over total wealth is $\frac{\rho x}{y}$ instead of $\frac{\rho}{y}$, and
- the growth rate becomes $g = \mu x + r(1-x) - \gamma - \frac{\rho x}{y}$.

Proceeding similarly to the analysis of the case without a risk-free asset, we obtain the following proposition, which clarifies that constrained allocations with and without a risk-free asset are very similar:

Proposition 15 *When there is a risk-free asset, the optimal controls (x, y, γ) are constant along the optimal trajectories ($a_t \equiv a$) and only depend on $(K, \mathbb{P})$ through the initial value a. Two cases are possible:*

1. *If the rate of return on the risk-free asset is relatively low, in the sense that*
$$r \leq \mu - \frac{\sigma^2}{\frac{\sigma^2}{\rho} y + \frac{1}{y}}, \tag{65}$$
where y is given by
$$\frac{\sigma^2}{\rho} y^3 + y = 1 - \frac{1}{v'(a)},$$
then there is no investment in the risk-free asset and the optimization problem is the same as in Proposition 8.

2. *If the rate of return on the risk-free asset is large enough that condition (65) does not hold, then the risk exposure y of the agents in the optimal contract is given implicitly by:*

$$r = \mu - \frac{\sigma^2}{\frac{\sigma^2}{\rho} y + \frac{1}{y}},$$

while the fraction of wealth invested in the risk asset is $x < 1$ such that

$$x[1 - \frac{1}{v'(a)}] = \frac{\sigma^2}{\rho} y^3 + y,$$

which implies that the optimal value of x increases in a.

This proposition illustrates the robustness of our results. The only difference introduced by the risk-free asset is a new control variable x, allowing the principal to maintain a constant exposure y of the agents to the risky asset, while adjusting to the agents' bargaining power a.

9 Conclusion

This paper extends Merton's (1969) analysis of consumption-savings and asset-allocation choices to the case in which the investor delegates the management of assets to many, privately informed, agents. The characteristics of the optimal incentive-constrained dynamic mechanism can be summarized as follows:

- Incentive compatibility implies that capital must be reallocated across agents as a function performance, despite the fact that on the equilibrium path performance only reflects luck. Moreover, in the optimal mechanism, capital reallocation is linear in relative performance.
- Agents are compensated with fees that are proportional to the amount of assets under their management.
- Exposing agents to their idiosyncratic risk increases the cost of holding risky capital and tilts the consumption-investment trade-off towards less investment. As a result, there is less capital accumulation than in the first best.
- Allocating capital as a function of performance leads to an increasingly heterogeneous distribution of capital among agents, which we characterize in quasi-closed form.

To obtain these results, we consider a continuum of agents and take a mean-field control approach to the investor's optimization problem. We thus develop a new methodology to solve dynamic contracting problems in which one of the state variables (the distribution of agents' utilities) is infinite-dimensional. This establishes a framework for a theory of large dynamic contracts, which complements the theory of large games. This new framework could be used to study incentive-constrained dynamic efficiency with heterogeneous agents in other contexts.

For example, our methodology could prove useful in dynamic macroeconomic models with heterogeneous agents and financial constraints (see Aiyagari (1994), Angeletos (2007), Achdou et al (2022), Bewley (1977), Brunnermeier and Sannikov (2014), Huguett (1997), and Krusell and Smith (1998).) Accounting for aggregate risk has proved mathematically challenging in this literature. To cope with this difficulty, papers have often relied on numerical approximations. For example, Krusell and Smith (1998) *assume* that agents' expectations only depend on a one-dimensional statistics and then numerically calibrate this "approximated equilibrium." The tools developed in the present paper could help obtain exact and more explicit solutions. This could help shed more light on the key economic mechanisms at play in these problems.

Another example is incentive-constrained dynamic learning and innovation. Our new methodology, which makes it possible to study contracting with many agents could be fruitful in this context, for which transiting from one to many agents raises important new economic issues.

- For example, Manso (2011) studies the dynamic interaction between a principal and a single agent who can discover some innovation. Extending this analysis to the case of many agents would be particularly interesting in a Schumpeterian context, in which a population of agents can innovate as in Aghion and Howitt (1992). In that environment, when one agent innovates, this makes previous technologies obsolete, but also makes future innovations more attractive. Our methodology could help study how this tradeoff interacts with dynamic incentives, and the efficiency properties of the optimal contract.

- Moreover, DeMarzo and Sannikov (2016), and Hoerner and Samuelson (2013), analyze dynamic principal-agent interactions under learning. The ability to compare the experiences of different agents could make learning more efficient, and at the same time affect the incentives of the agents. In this context, our methodology could be useful to study the dynamic efficiency of incentive constrained allocations.

- Jenter and Lewellen (2021) study how CEO turnover interacts with incentives. In a model with many CEOs, CEO turnover could arise naturally, as better performing CEOs would be reallocated the assets of worse performing CEOs. Our methodology could be used to study the incentive-constrained efficiency of such dynamic reallocations.

While such analyses might be quite challenging, we hope the methodological tools developed in this paper could help address these questions.

Appendix A: Proofs and propositions not provided in the main text

Proof of Proposition 2

Step 1: Guess Inspired by the Merton Model, we begin by calculating the principal's value associated with the guessed controls:

$$c_t^P = \gamma^P K_t, k_t^i = K_t, c^i = \gamma^i K_t, \text{ for all } i \in \{1, \dots, N\}. \tag{66}$$

Substituting these controls in the resource constraint (5) yields the dynamics of capital:

$$\frac{dK_t}{K_t} = \left(\mu - \gamma^P - \frac{1}{N}\sum_i \gamma^i\right) dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_i dZ_t^i. \tag{67}$$

Thus K_t follows a log-normal distribution and

$$\mathbb{E}(\log K_t) = \log K + (g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N})t, \tag{68}$$

where $g = \mu - \gamma^P - \frac{1}{N}\sum_i \gamma^i$. An integration gives the principal's value function in the case of logarithmic utility

$$\mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \log(\gamma^P K_t) dt\right] = \log(\gamma^P K) + \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right). \tag{69}$$

Using (6), the continuation pay-off of agent i with logarithmic utility is given by

$$\omega_t^i = \log(\gamma^i K_t) + \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right), \tag{70}$$

implying that the ratio $\frac{\exp \omega_t^i}{K_t}$ remains constant along optimal trajectories. This property implies in turn that risk is equally shared across agents when the controls are given by (66). Indeed, by differentiation of (70) we obtain:

$$d\omega_t^i = d(\log K_t) = (g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N})dt + \sigma_A dZ_t^A + \frac{\sigma}{N}\sum_j dZ_t^j, \tag{71}$$

which yields Equation (3).

We deduce from (70) that

$$\frac{A_t}{K_t} = \gamma \exp \frac{1}{\rho}\Big(g - \frac{1}{2}\Big(\sigma_A^2 + \frac{\sigma^2}{N}\Big)\Big),$$

where

$$\gamma = \frac{1}{N}\sum_{i=1}^N \gamma^i.$$

Therefore, when focusing on the controls given by (66), the principal face the static optimization problem

$$\sup_{\gamma^P,\gamma} \log(\gamma^P K) + \frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right)$$

under the constraint

$$b = \frac{A_0}{K_0} = \gamma \exp\left(\frac{1}{\rho}\left(g - \frac{\sigma_A^2}{2} - \frac{\sigma^2}{2N}\right)\right).$$

The feasible set is given by $b < m_N$ where m_N is given by (11). Solving the static optimization problem gives the principal value under the controls (66)

$$V(K, \mathbb{W}) = \log(K) + \log(m_N - b). \tag{72}$$

To finalize the analysis of the implications of our guess, we show that it yields the aggregate consumption equation (19). Equations (69) and (70) imply that the Pareto frontier, i.e., the sum of the value of the principals and the average value of the agents, is equal to

$$(\gamma^P + \frac{1}{N}\sum_i \exp\omega_i) K \exp\left(\frac{1}{\rho}\left[g - \frac{1}{2}(\sigma_A^2 - \frac{\sigma^2}{N})\right]\right).$$

Maximising over g yields

$$\mu - g = \rho,$$

which is (19).

Step 2: Verification The next step completes our guess and verify approach by establishing that the solution obtained under our guess, and stated in equation (72), is a solution to the HJB equation (10). The partial derivatives of V are

$$V_K = \frac{m}{K(m-b)}, V_{KK} = -\frac{m^2}{K^2(m-b)^2}, V_{\omega_i} = -\frac{\exp\omega_i}{NK(m-b)},$$

$$V_{\omega_i K} = \frac{m\exp\omega_i}{NK^2(m-b)^2}, V_{\omega_i\omega_i} = -\frac{\exp 2\omega_i}{N^2K^2(m-b)^2} - \frac{\exp\omega_i}{NK(m-b)},$$

and, for $j \neq i$,

$$V_{\omega_i\omega_j} = -\frac{\exp\omega_i \exp\omega_j}{N^2K^2(m-b)^2}.$$

We plug the conjectured value function V in (10) to obtain

$$\rho\log K + \rho\log(m_N - b) =$$

$$\sup_X \left\{\rho\log c^P - \sum_{i=1}^N \rho(\omega_i - \log(c^i))\frac{\exp\omega_i}{NK(m_N-b)} + \frac{m_N}{K(m_N-b)}\Big(\mu K - c^P - \frac{1}{N}\sum_{i=1}^N c^i\Big)\right.$$

$$+\sum_{i=1}^{N}\Big(\sigma_A^2 K\beta^{A,i}+\frac{\sigma^2}{N^2}\sum_{j=1}^{N}k^j\beta^{ij}\Big)\frac{m_N\exp\omega_i}{NK^2(m_N-b)^2}$$

$$-\frac{1}{2}\Big(\sigma_A^2K^2+\frac{\sigma^2}{N^2}\sum_{j=1}^{N}(k^j)^2\Big)\frac{m_N^2}{K^2(m_N-b)^2}$$

$$-\frac{1}{2}\sum_{j\neq i}\Big(\sigma_A^2\beta^{A,i}\beta^{A,j}+\frac{\sigma^2}{N^2}\sum_{l=1}^{N}\beta^{il}\beta^{jl}\Big)\frac{\exp\omega_i\exp\omega_j}{N^2K^2(m_N-b)^2}$$

$$-\frac{1}{2}\sum_{i=1}^{N}\Big(\sigma_A^2(\beta^{A,i})^2+\frac{\sigma^2}{N^2}\sum_{l=1}^{N}(\beta^{il})^2\Big)\Big(\frac{\exp 2\omega_i}{N^2K^2(m-b)^2}+\frac{\exp\omega_i}{NK(m_N-b)}\Big)$$

$$+\lambda\Big(\sum_{i=1}^{N}k^i-K\Big)\Big\}. \tag{73}$$

The first order conditions give:

- $\frac{\rho}{c^P}=\frac{m_N}{K(m_N-b)}$ implying $c^P=\rho(1-\frac{b}{m_N})K$,
- $\frac{\rho}{c^i}\frac{\exp\omega_i}{NK(m_N-b)}=\frac{m}{NK(m_N-b)}$ implying $c^i=\frac{\rho\exp\omega^i}{m_N}$,
- $\beta^{ij}=\beta^{A,i}=1$ for all i,j
- $k^i=K$, for all i.

Equation (73) rewrites

$$\begin{aligned}\rho\log(K)+\rho\log(m_N-b)&=\rho\log(\frac{\rho}{m_N})+\rho\log(m_N-b)+\rho\log(K)\\&+\sum_{i=1}^{N}\rho\log(\frac{\rho}{m_N})\frac{\exp\omega_i}{NK(m_N-b)}\\&+\frac{m}{K(m_N-b)}\Big(\mu K-\rho(1-\frac{b}{m_N})K-\frac{\rho}{m_N}bK)\Big)\\&+\sum_{i=1}^{N}(\sigma_A^2K+\frac{\sigma^2}{N}K)\frac{m_N\exp\omega_i}{NK^2(m_N-b)^2}\\&-\frac{1}{2}(\sigma_A^2K^2+\frac{\sigma^2}{N}K^2)\frac{m_N^2}{K^2(m_N-b)^2}\\&-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\sigma_A^2+\frac{\sigma^2}{N})\frac{\exp\omega_i\exp\omega_j}{N^2K^2(m_N-b)^2}\\&-\frac{1}{2}\sum_{i=1}^{N}(\sigma_A^2+\frac{\sigma^2}{N})\frac{\exp\omega_i}{NK(m_N-b)}.\end{aligned}$$

After simplifications, we obtain

$$0 = \frac{m_N}{m_N - b}\Big(\rho \log(\frac{\rho}{m_N}) + \mu - \rho - \frac{1}{2}\big(\sigma_A^2 + \frac{\sigma^2}{N}\big)\Big),$$

which is true by the definition of m given in (11).
To confirm that our guess represents the optimal contract, we must verify the transversality condition. Let us denote by π the controls $(k, c^P, c, \beta, \beta_A)$ to alleviate the notation. For $\varepsilon > 0$, let us define the set

$$\mathcal{A}_\varepsilon = \{X \text{ such that } A_t^{(X)} \leq (m - \varepsilon) K_t^{(X)} \text{ for } t \geq 0\},$$

where A_t^X and K_t^X are respectively the average inverse utility and the capital processes under the control policy X and the set $\mathcal{A}$ as the union of $\mathcal{A}_\varepsilon$. We will prove that for every control $X \in \mathcal{A}$, we have the transversality condition $\lim_{t\to+\infty} e^{-\rho t}\left(\log K_t^X + \log(m_N - b_t^X)\right) = 0$ where

$$b_t^X = \frac{A_t^{(X)}}{K_t^{(X)}}.$$

Take $X \in \mathcal{A}$, then there is $\varepsilon > 0$, such that $X \in \mathcal{A}_\varepsilon$ and $\log(m_N - b_t^X) \leq \log(\frac{1}{\varepsilon})$. On the other hand, because the consumption processes are non-negative, the capital process $K^{(X)}$ given in (5) is dominated by the Geometric Brownian motion.

QED

Proof of Proposition 3

The convergence results established in Appendix C-1 facilitate a direct proof of Proposition 3. Specifically, the value function characterized in (13) is shown to be a jointly continuous mapping, $v(m_n, K, \mathbb{P}^{(N)})$, where $\mathbb{P}^{(N)}$ is the empirical measure. As the number of agents N tends to infinity, this sequence of value functions converges uniformly to the limit value function:

$$v(m_\infty, K, \mathbb{P}) = \log(m_\infty K - A). \tag{74}$$

This limiting form confirms that the functional structure of the finite-economy value function is preserved in the continuum for the first-best. The result demonstrates that the complex interactions of the N-agent case collapse into a tractable analytical form, where the aggregate state is summarized by the effective capital stock and some statistics A of the underlying measure $\mathbb{P}$.

Proof of Proposition 5

With the change of control variables: $c^P = \gamma K$, and $c = \frac{\rho K}{y}$, the Hamilton Jacobi Bellman equation (24) becomes:

$$\rho v(a) = \sup_{\gamma, \beta \geq y, \beta^A} \rho \log \gamma + \rho v'(a)(a - \log \frac{\rho}{y}) + (1 - v'(a))(\mu - \gamma - \frac{\rho}{y})$$

$$-v''(a)[\beta^A\sigma_A^2+\beta\sigma^2-\frac{1}{2}\left(\sigma_A^2(\beta^A)^2+\sigma^2\beta^2\right)]-\frac{v''(a)+v'(a)-1}{2}\left[\sigma^2+\sigma_A^2\right]. \tag{75}$$

Assuming that $v''(a)<0$ (which we checked numerically), the right hand side of this equation is a concave function of β^A and β. It is therefore maximum for $\beta^A=1$. Moreover, the incentive compatibility condition is always binding: $\beta\equiv y$. Indeed, if it were not binding, the first order conditions would imply $\beta=1$ and $y=1-\frac{1}{v'(a)}$. Since $v'(a)<0$, this would violate the constraint $\beta\geq y$. If we plug the optimal values $\beta^A=1$ and $\beta=y$ in the above equation, we obtain the HJB equation given in the proposition.

QED

Proof of Proposition 6

Using our guess (42, 44, and 43) and the corresponding expressions of the derivatives and gradients of V (31, 45, 46, 47, and 48), the Hamilton Jacobi Bellman equation (39) rewrites as follows (for brevity we omit the arguments K and $\mathbb{P}$ in γ, $y(\omega)$, and $k(\omega)$):

$$\rho v(a)=\sup_{\gamma,y(.),\beta^A(.),k(.)}\rho\log\gamma+[1-v'(a)][\mu-\gamma-\int\frac{\rho k(\omega)}{Ky(\omega)}d\mathbb{P}(\omega)]+\frac{\sigma_A^2}{2}(v''(a)+v'(a)-1)$$

$$+\int[\rho(\omega-\log\frac{\rho k(\omega)}{y(\omega)})+\frac{\sigma^2y^2(\omega)+\sigma_A^2\beta_A^2(\omega)}{2}]\frac{v'(a)\exp\omega}{A}d\mathbb{P}(\omega)$$

$$-\int[v''(a)\frac{\exp\omega}{A}\sigma_A^2\beta_A(\omega)]d\mathbb{P}(\omega)$$

$$+\frac{\sigma_A^2[v''(a)-v'(a)]}{2}[\int\frac{\exp\omega}{A}\beta_A(\omega)d\mathbb{P}(\omega)]^2.$$

The first order conditions with respect to $y(\omega)$ is

$$\frac{[1-v'(a)]\rho k(\omega)}{Ky^2(\omega)}+[\frac{\rho}{y(\omega)}+\sigma^2y(\omega)]\frac{v'(a)\exp\omega}{A}=0, \tag{76}$$

while the first order condition with respect to $k(\omega)$ is

$$\frac{\rho[1-v'(a)]}{Ky(\omega)}+\frac{\rho v'(a)\exp\omega}{Ak(\omega)}=0. \tag{77}$$

(76) implies that

$$\frac{\rho v'(a)\exp\omega}{Ak(\omega)}=-\frac{[1-v'(a)]\rho}{K[y(\omega)+\frac{\sigma^2y^3(\omega)}{\rho}]}.$$

Inserting this expression into (77) gives:

$$\frac{\rho[1-v'(a)]}{Ky(\omega)}-\frac{[1-v'(a)]\rho}{K[y(\omega)+\frac{\sigma^2y^3(\omega)}{\rho}]}=0,$$

which implies that $y(\omega)$ is necessarily independent of ω.

Since $c(\omega) = \rho k(\omega)/y(\omega)$, that $y(\omega)$ is independent of ω implies that the capital allocated to an agent with continuation utility ω is proportional to $\exp \omega$. The capital allocation constraint determines the proportionality constant:

$$k(\omega) = K \frac{\exp \omega}{A}.$$

Moreover, the first order condition with respect to γ yields

$$\gamma = \frac{\rho}{1 - v'(a)}.$$

Finally, the first order condition with respect to $\beta_A(\omega)$ implies that

$$\beta_A(\omega)v'(a) - v''(a) = [v'(a) - v''(a)][\int \frac{\exp \omega'}{A} \beta_A(\omega') d\mathbb{P}(\omega')].$$

Since $v''(a) < 0$, this implies that $\beta_A \equiv 1$.

QED

Proof of Proposition 7

According to Proposition 6, we have

- $a_t = \log A_t - \log K_t$ with $A_t = \int \exp(w) dP_t(\omega)$
- $k(\omega_t) = e^{-a_t} e^{\omega_t}, \quad \beta_A = 1$
- $c(\omega_t) = \frac{\rho}{y(a_t)} k(\omega_t) \quad c_t^P = \gamma(a_t) K_t.$

The implies that the dynamics of the continuation utility is

$$dw_t = \rho \left(a_t - \log \frac{\rho}{y(a_t)} \right) dt + \sigma \rho y(a_t) dZ_t + \sigma_A dZ_t^A \tag{78}$$

Itô's formula gives

$$\exp(w_t) = \exp(w_0) + \int_0^t \exp(w_s) \left(\rho \left(a_s - \log \frac{\rho}{y(a_s)} \right) + \frac{\sigma^2 \rho^2 y^2(a_s) + \sigma_A^2}{2} \right) ds$$
$$+ \int_0^t \sigma \rho y(a_s) \, dZ_s + \sigma_A Z_t^A.$$

Taking the conditional expectations with respect to the filtration generated by Z^A, we get

$$dA_t = A_t \left(\left(\rho \left(a_t - \log \frac{\rho}{y(a_t)} \right) + \frac{\sigma^2 \rho^2 y^2(a_t) + \sigma_A^2}{2} \right) dt + \sigma_A dZ_t^A \right) \tag{79}$$

So

$$da_t = \frac{dA_t}{A_t} - \frac{dK_t}{K_t} = \left[\rho \left(a_t - \log \frac{\rho}{y(a_t)} \right) + \frac{\sigma^2 \rho^2 y^2(a_t) + \sigma_A^2}{2} - \left(\mu - \frac{\rho}{y(a_t)} - \gamma(a_t) \right) \right] dt \tag{80}$$

Differentiating (52) along the optimal policy, we get for all $a > 0$,

$$0 = v''(a) \left[a - \log \frac{\rho}{y(a)} - \frac{1}{\rho} \left(\mu - \gamma(a) - \frac{\rho}{y(a)} - \frac{\sigma_A^2 + \sigma^2 y^2(a)}{2} \right) \right], \qquad (81)$$

which yields $da_t = 0$ along the optimal policy.

QED

Proof of Proposition 8

Note that γ and y must be positive for the log expressions in (53) and (54) to be well defined. The feasible set is thus bounded. It is non empty if and only if

$$a < a_{max} \equiv \sup_y [\log \frac{\rho}{y} + \frac{1}{\rho}(\mu - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2})]$$

In this case, the solution exists and the value function equals the maximum of the Lagrangian:

$$v(a) = \sup_{\gamma,y} \Big(\log \gamma + \frac{1}{\rho}(\mu - \frac{\sigma_A^2}{2} - \gamma - \frac{\rho}{y}) + \nu[a - \log \frac{\rho}{y} - \frac{\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2}}{\rho}] \Big).$$

By the envelope theorem, $v'(a) = \nu$ and we can write

$$v(a) = \sup_{\gamma,y} \Big(\log \gamma + \frac{1}{\rho}(\mu - \frac{\sigma_A^2}{2} - \gamma - \frac{\rho}{y}) + v'(a)[a - \log \frac{\rho}{y} - \frac{\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2}}{\rho}] \Big),$$

which coincides with the Hamilton Jacobi Bellman equation (52).
It remains to prove that v is concave. We define the objective function $f(\gamma, y)$ and the constraint function $g(\gamma, y)$ as follows:

$$f(\gamma, y) = \log \gamma + \frac{1}{\rho} \left(\mu - \frac{\sigma_A^2}{2} - \gamma - \frac{\rho}{y} \right)$$

$$g(\gamma, y) = \log \frac{\rho}{y} + \frac{1}{\rho} \left(\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2 + \sigma^2 y^2}{2} \right)$$

The value function is defined as:

$$v(a) = \sup_{\gamma,y} \{ f(\gamma, y) \mid g(\gamma, y) = a \}$$

We define the Lagrangian L by

$$L(\gamma, y, \nu) = f(\gamma, y) + \nu \left(a - g(\gamma, y) \right).$$

By the Envelope Theorem, we have already observed that the derivative of the value function is the Lagrange multiplier:

$$v'(a) = \frac{\partial L}{\partial a} = \nu$$

To show $v(a)$ is concave, we must show $v''(a) = \frac{d\nu}{da} \le 0$.
The First-Order Condition (FOC) with respect to γ is:

$$\frac{\partial L}{\partial \gamma} = \frac{1}{\gamma} - \frac{1}{\rho} - \nu\left(-\frac{1}{\rho}\right) = 0 \implies \frac{1}{\gamma} = \frac{1-\nu}{\rho} \implies \gamma = \frac{\rho}{1-\nu}$$

The FOC with respect to y is:

$$\frac{\partial L}{\partial y} = \frac{1}{y^2} - \nu\frac{\partial g}{\partial y} = 0$$

We also have,

$$\frac{\partial g}{\partial y} = -\frac{1}{y} + \frac{1}{y^2} - \frac{\sigma^2 y}{\rho} = -\frac{\rho y - \rho + \sigma^2 y^3}{\rho y^2}$$

Let $D(y) = \rho(y-1) + \sigma^2 y^3$. Then $\frac{\partial g}{\partial y} = -\frac{D(y)}{\rho y^2}$. Substituting this into the FOC for y:

$$\frac{1}{y^2} - \nu\left(-\frac{D(y)}{\rho y^2}\right) = 0 \implies 1 + \frac{\nu D(y)}{\rho} = 0 \implies \nu = -\frac{\rho}{D(y)}$$

Since $\gamma > 0$ and $\rho > 0$, we require $1-\nu > 0$, so $\nu < 1$.Substituting $\nu = -\rho/D$:

$$\gamma = \frac{\rho}{1+\rho/D} = \frac{\rho D}{D+\rho}$$

Since $\rho > 0$ and $D + \rho = \rho y + \sigma^2 y^3 > 0$ (for $y > 0$), we must have $D(y) > 0$ for a valid solution $\gamma > 0$. This implies $\nu = -\rho/D < 0$, so $v'(a) < 0$.

We use the chain rule: $v''(a) = \frac{d\nu}{dy} / \frac{da}{dy}$.
Since $\nu = -\rho/D(y)$, and $D'(y) = \rho + 3\sigma^2 y^2 > 0$:

$$\frac{d\nu}{dy} = \frac{\rho D'(y)}{D(y)^2} > 0$$

From the constraint $a = g(\gamma(y), y)$, we differentiate with respect to y:

$$\frac{da}{dy} = \frac{\partial g}{\partial y} + \frac{\partial g}{\partial \gamma}\frac{d\gamma}{dy}$$

We know $\frac{\partial g}{\partial y} = -\frac{D}{\rho y^2} < 0$ and $\frac{\partial g}{\partial \gamma} = -\frac{1}{\rho} < 0$.From $\gamma = \frac{\rho D}{D+\rho}$, the derivative $\frac{d\gamma}{dy} = \frac{\rho^2 D'}{(D+\rho)^2} > 0$.Thus, $\frac{da}{dy} < 0$.
Finally,

$$v''(a) = \frac{d\nu/dy}{da/dy} < 0$$

Since the second derivative is strictly negative, the value function $v(a)$ is strictly concave.

QED

Proof of Proposition 10

To apply the verification theorem (Proposition 17) and conclude that the function $\log(K) + v(a)$ coincides with the principal value, it remains to prove that $\mathbb{E}\left[e^{-\rho t}\left(\log(K_t) + v(\frac{A_t}{K_t})\right)\right]$ converges to 0 when t goes to ∞ for all controls for which the ratio $\frac{A_t}{K_t}$ is strictly positive and strictly lower than a_{max}. Denote by X the controls $(k, c^P, c, \beta, \beta_A)$. For $\varepsilon > 0$, let us define the set

$$\mathcal{A}_\varepsilon = \{X \text{ such that } \varepsilon K_t^{(X)} \leq \int \exp(\rho\omega) d\mathbb{P}_t^{(X)}(\omega) \leq (a_{max} - \varepsilon) K_t^{(X)} \text{ for } t \geq 0\}$$

and the set $\mathcal{A}$ as the union of $\mathcal{A}_\varepsilon$. We will prove that for every control $X \in \mathcal{A}$, we have the transversality condition $\lim_{t \to +\infty} e^{-\rho t}\left(\frac{\log K_t^X}{\rho} + v(a_t^X)\right) = 0$ where

$$a_t^X = \frac{\int \exp(\rho\omega) d\mathbb{P}_t^{(X)}(\omega)}{K_t^{(X)}}.$$

Take $X \in \mathcal{A}$. There is $\varepsilon > 0$, such that $X \in \mathcal{A}_\varepsilon$. Because v is continuous, it is bounded by a constant C_ε on the interval $[\varepsilon, a_{max} - \varepsilon]$, and we have

$$e^{-\rho t} v(a_t^X) \leq e^{-\rho t} C_\varepsilon.$$

On the other hand, because the consumption processes are non-negative, the capital process $K^{(X)}$ given in (36) is dominated by the Geometric Brownian motion

$$K \exp[(\mu - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A].$$

Consequently, we have

$$\mathbb{E}\left[e^{-\rho t} \log(K_t)\right] \leq \mathbb{E}\left[e^{-\rho t}(\log(K) + (\mu - \frac{\sigma_A^2}{2})t + \sigma_A Z_t^A)\right] \text{ which converges to 0.}$$

Proof of Proposition 12

By Proposition 8, μ and γ maximize the Lagrangian

$$\log\gamma + \frac{1}{\rho}\left(\mu - \gamma - \frac{\rho}{y} - \frac{\sigma_A^2}{2}\right) + \lambda\left[\log\frac{\rho}{y} + \frac{1}{\rho}\left(\mu - \gamma - \frac{\rho}{y}\right) - \frac{\sigma_A^2 + \sigma^2 y^2}{2}\right]$$

The first order condition with respect to γ yields

$$\gamma = \frac{\rho}{1 + \lambda}.$$

The first order condition with respect to y yields

$$y + \frac{\sigma^2}{\rho} y^3 = \frac{1 + \lambda}{\lambda}.$$

Combining the two yields,

$$g_{SB} = \mu - \gamma - \frac{\rho}{y},$$

and consequently (58). Finally, combining g_{SB} with Proposition 6 yields (57) and (59).

QED

Proof of Proposition 14

Our first step is to consider a risk-neutral principal facing a single agent. In that case, we show that, when $\mu \geq \rho + \frac{\sigma^2+\sigma_A^2}{2}$, the principal's value function is infinite. To see this, consider the following incentive compatible contract

$$c_t = \eta K_t, \beta_t = \frac{\rho}{\eta}, \beta_A = 1,$$

where η is a constant, to be optimally chosen by the principal. The principal consumes zero until date T, and the entire capital stock K_T at date T.[23] Along this specific contract, the principal value is

$$\sup_{\eta} \mathbb{E}\left[e^{-\rho T} K_T\right]$$

under the participation constraint

$$\omega = \mathbb{E}\left[\int_0^T \rho e^{-\rho t} \log(\eta K_t)\, dt\right] + e^{-\rho T}\omega, \tag{82}$$

where we assume that at time T the agents revert to their initial outside option, which they value at ω. As T goes to infinity, the right-hand side of (82) converges to

$$\log K + \log \eta + \frac{1}{\rho}\left(\mu - \eta - \frac{\sigma^2 + \sigma_A^2}{2}\right).$$

Observe that the function $\eta \mapsto \log \eta + \frac{1}{\rho}\left(\mu - \eta - \frac{\sigma^2+\sigma_A^2}{2}\right)$ is a concave function, which reaches its maximum at $\eta = \rho$. Denote by $\log m_1$ the maximum value of this function. For $a = \omega - \log K < \log m_1$, there exists $\eta_T \leq \rho$ that satisfies the participation constraint (82) for large T. Moreover, the principal value dominates

$$\begin{aligned}\mathbb{E}\left[e^{-\rho T} K_T\right] &= K \exp\left(\left(\mu - \eta_T - \frac{\sigma^2 + \sigma_A^2}{2}\right)T\right) \\ &\geq K \exp\left(\left(\mu - \rho - \frac{\sigma^2 + \sigma_A^2}{2}\right)T\right),\end{aligned}$$

[23] We impose the plausible condition that consumption is non-negative. In the contract we describe, this constraint is binding until time T. If we did not impose this constraint, the principal could achieve an even higher utility, by having large negative consumption at time 0, in order to increase investment in the productive asset.

which converges to $+\infty$ under the assumption

$$\mu \geq \rho + \frac{\sigma^2 + \sigma_A^2}{2}. \tag{83}$$

We conclude that, under (83), the principal can obtain an infinite value with a contract for which the incentive and participation constraints of the agents hold.

Our second step is to extend the result to the case in which the principal faces a continuum of agents. In that case, let V_∞ denote the principal's value function. We assume that the agent' utility distribution is given in time 0 by the probability distribution $\mathbb{P}$. The principal distributes capital to the agents at $t = 0$ and then allows the capital of each agent to evolve independently (i.e., no capital reallocation occurs between agents after time 0), and implements the optimal single agent contract described in the first step of the present proof. Since this contract is a priori suboptimal, the following inequality holds:

$$V_\infty(K, \mathbb{P}) \geq \int V(K, \omega)\mathbb{P}(d\omega),$$

where $V(K, \omega)$ is the value of the risk-neutral principal with a single agent. Since we showed in the first step of the proof that this risk-neutral principal's value is infinite, we can conclude that the principal can obtain infinite utility with a continuum of agents.

QED

Proof of Proposition 15

The new expression of the reduced HJB equation is

$$\rho v(a) = \sup_{x \leq 1, y, \gamma} \rho \log \gamma + (1 - v'(a))[\mu x + r(1-x) - \gamma - \frac{\rho x}{y}]$$
$$+ v'(a)(\rho a - \rho \log \frac{\rho x}{y} + \frac{\sigma^2 y^2}{2}).$$

The first order conditions give

- $\gamma = \frac{\rho}{1 - v'(a)}$,
- $(1 - v'(a))(\mu - r - \frac{\rho}{y}) \geq v'(a)\frac{\rho}{x}$ (with equality if $x < 1$) and
- $(1 - v'(a))\frac{\rho x}{y^2} + v'(a)(\frac{\rho}{y} + \sigma^2 y) = 0$.

Two cases are possible:

- The risk-free bond is not used ($x = 1$) and we are back to the previous case where

  $$\frac{\sigma^2}{\rho} y^3 + y = 1 - \frac{1}{v'(a)}.$$

  This happens when

  $$r \leq \mu - \frac{\rho}{y} + \frac{\rho}{\frac{\sigma^2}{\rho} y^3 + y}.$$

- The solution is interior ($x < 1$) and

$$x(1 - \frac{1}{v'(a)}) = \frac{\sigma^2}{\rho} y^3 + y.$$

  Inserting this expression into the first order condition with respect to x (which becomes an equality) and rearranging terms, we find that y must be a solution of the equation

$$\frac{\sigma^2}{\mu - r} = \frac{\sigma^2}{\rho} y + \frac{1}{y}.$$

Thus, as expected, the risk-free asset is only used when r is large enough. In this case, y depends only on r but not on a, while x increases in a. When the bargaining power of the agent increases, the agent's risk exposure remains constant while the adjustment is made through the proportion x of total wealth that is invested in risky assets.

$$\frac{\sigma^2}{\rho} y^3 + y = x < 1.$$

Now, without a risk-free asset the optimal value of y was the root of

$$\frac{\sigma^2}{\rho} y^3 + y = \frac{1 + \lambda}{\lambda} > 1.$$

So the optimal value of y when there is a risk-free asset is lower than its counterpart when there is no risk-free asset.

QED

Proposition 16 *If i) for every Lagrange multiplier process one can find an optimal control X_λ such that*

$$V_\lambda = \int_0^\infty e^{-\rho t} \left(\log c_\lambda^P + \lambda(K_t, \mathbb{P}_t) \left(K_t - \int k_\lambda(., \omega) d\mathbb{P}_t(\omega) \right) \right) dt$$

and ii) there exists a Lagrange multiplier λ_0 such that for all $t \geq 0$,

$$K_t = \int k_{\lambda_0}(., \omega) d\mathbb{P}_t(\omega),$$

then $V = V_{\lambda_0}$ and X_{λ_0} solves the principal problem.

Proof of Proposition 16 Let X be an admissible feedback control. We denote

$$J_\lambda^X = \int_0^\infty e^{-\rho t} \left(\log c_t^P + \lambda(K_t^{(X)}, \mathbb{P}_t^{(X)}) \left(K_t^{(X)} - \int k_\lambda(., \omega) d\mathbb{P}_t^{(X)}(\omega) \right) \right) dt,$$

and

$$J^X = \int_0^\infty e^{-\rho t} \log(c_t^P) \, dt.$$

For every Lagrange multiplier λ, we have by assumption i),

$$V_\lambda = J_\lambda^{X_\lambda} \geq J_\lambda^X.$$

In particular, for $\lambda = \lambda_0$, $V_{\lambda_0} = J_{\lambda_0}^{X_{\lambda_0}} = J^{X_{\lambda_0}}$, because the constraint is binding for the Lagrange multiplier λ_0.
On the other hand, for every control X that binds the constraint, we have $V_{\lambda_0} = J_{\lambda_0}^{X_{\lambda_0}} \geq J_{\lambda_0}^X = J^X$ yielding $V_{\lambda_0} \geq \sup_{X \text{ binds}} J^X$=V.
Because X_{λ_0} binds the constraint, we have $V = V_{\lambda_0}$ and the proof is complete.
QED

Substituting β from (40), the Hamilton Jacobi Bellman equation associated with the control problem (41) is

$$\rho V(K,\mathbb{P}) = \sup_{c,c^P,\beta^A,k} \quad \rho u(c^P) + \left(\mu K - c^P - \int c(\omega) d\mathbb{P}(\omega)\right) V_K + \frac{\sigma_A^2 K^2}{2} V_{KK}$$

$$+ \int [\rho(\omega - u(c(\omega)))\partial_\omega \nabla V(\omega) + \frac{(\sigma\rho k u'(c))^2(\omega) + (\sigma_A \beta_A)^2(\omega)}{2} \partial^2_{\omega\omega} \nabla V(\omega)] d\mathbb{P}(\omega)$$

$$+ \int [\partial^2_{\omega K} \nabla V(\omega) \sigma_A^2 \beta_A(\omega) K - \lambda(k(\omega) - K)] d\mathbb{P}(\omega)$$

$$+ \frac{\sigma_A^2}{2} \int\int \partial^2_{\omega\omega'} \nabla^2 V(\omega,\omega') \beta_A(\omega)\beta_A(\omega') d\mathbb{P}(\omega) d\mathbb{P}(\omega'). \tag{84}$$

Inspired by classical verification theorems for stochastic control of diffusion processes, we have the following verification theorem.

Proposition 17 *(Verification Theorem) Let $\lambda(.)$ be a given Lagrange multiplier, and $v^\lambda(K,\mathbb{P})$ be a L-differentiable function as defined in Definition 20. If*

i) v^λ is a solution to (84) with the transversality condition[24]

$$\lim_{t\to+\infty} \mathbb{E}\left[e^{-\rho t} v^\lambda(K_t,\mathbb{P}_t)\right] = 0,$$

ii) and there exists a control X_λ^* that reaches the maximum in (84),

then v^λ coincides with V_λ defined by (41). *Moreover, if there is a Lagrange multiplier λ_0 such that for every $t \geq 0$,*

$$\int k_{\lambda_0}(\omega) d\mathbb{P}_t(\omega) = K_t$$

then v^{λ_0} is the principal value function.

[24] If v^λ is a solution to (92), the transversality condition writes

$$\lim_{t\to+\infty} e^{-\rho t} v^\lambda(K_t,\mathbb{P}_t) = 0$$

Proof of Proposition 17 We make the proof for $\sigma_A = 0$. Fix $\mathbb{P} \in \mathbb{P}_2(\mathbb{R})$ and a Lagrange multiplier λ. Let $\mathbb{P}_t$ be the probability distribution of the random variable ω_t when the initial probability distribution of ω_0 is $\mathbb{P}$. Let us consider some arbitrary feedback control $X(K_t, \mathbb{P}_t, \omega_t)$. To alleviate notations, we omit the dependence in the control and denote K_t and $\mathbb{P}_t$ the distribution along the control X.
We apply Itô's formula (92) to $v^\lambda(K_t, \mathbb{P}_t)$ between $s = 0$ and $s = t$ for $t > 0$.

$$
\begin{aligned}
e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t) = v^\lambda(K, \mathbb{P}) \\
&+ \int_0^t e^{-\rho s} \Big(-\rho v^\lambda(K_s, \mathbb{P}_s) \\
&+ v_K^\lambda(K_s, \mathbb{P}_s) \left(\mu K_s - c^P(K_s, \mathbb{P}_s) - \int c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \Big) \, ds \\
&+ \int_0^t e^{-\rho s} \int \partial_\omega \nabla v^\lambda [K_s, \mathbb{P}_s](\omega)(\rho\omega - \log c(K_s, \mathbb{P}_s, \omega)) d\mathbb{P}_s(\omega) \, ds \\
&+ \int_0^t e^{-\rho s} \int \partial_{\omega\omega} \nabla v^\lambda [(K_s, \mathbb{P}_s](\omega) \frac{\sigma^2}{2} y^2(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \, ds.
\end{aligned}
$$

We deduce from the Bellman equation satisfied by v^λ that

$$
\begin{aligned}
v^\lambda(K, \mathbb{P}) \geq e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t) \\
&+ \int_0^t e^{-\rho s} \left(\log(c^P(K_s, \mathbb{P}_s) + \lambda(K_s, \mathbb{P}_s) \left(K_s - \int y(K_s, \mathbb{P}_s, \omega) c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds.
\end{aligned}
$$

Letting t tend to $+\infty$ and using the transversality condition, we obtain

$$
v^\lambda(K, \mathbb{P}) \geq \int_0^\infty e^{-\rho s} \left(\log(c^P(K_s, \mathbb{P}_s)) + \lambda(K_s, \mathbb{P}_s) \left(K_s - \int y(K_s, \mathbb{P}_s, \omega) c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds = J_\lambda^X.
$$

Since the control X is arbitrary, we obtain $v^\lambda(K, \mathbb{P}) \geq V_\lambda$. On the other hand, let us apply the same Itô argument with the control X_λ^* to obtain

$$
v^\lambda(K, \mathbb{P}) = J_\lambda^{X_\lambda^*} \leq V_\lambda,
$$

which yields that $v^\lambda = V_\lambda$. We conclude the proof by applying Proposition 16. QED

Appendix B: Numerical analysis of the one-agent problem

The following lemma provides a precise specification of the boundary conditions that are needed for the numerical solution.

Lemma 18 *The function v satisfies both $\lim_{a\to-\infty} v(a) = \log m_1$ where m_1 is given by* (11) *with $N = 1$ and $\lim_{a\to\log m_1} v(a) = -\infty$.*

Proof of Lemma 18:

We first prove the limit as $a \to -\infty$. Let $K_0 > 0$ be some capital level and assume that the principal's consumption is zero. This lack of principal consumption removes the agent's incentive for private diversion, leading to a situation that resolves into a classical Merton problem

$$\omega(K_0) = \sup_c \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t}\log(c_t)\mathrm{d}t\right].$$

It is well-known that the optimal consumption in this setting is given by

$$\hat{c}_t = \rho K_t,$$

for all $t \geq 0$. This leads to

$$\omega(K_0) = \log(K_0) + \log(m_1).$$

Indeed, the dynamics of K, cf. Eq. (5), under this optimal consumption strategy implies that

$$K_t = K_0 \exp\left(\left(\mu - \rho - \frac{\sigma^2+\sigma_A^2}{2}\right)t + \sigma_A Z_t^A + \sigma Z_t\right).$$

Thus, using Fubini's Theorem, we find that

$$\begin{aligned}
\omega(K) &= \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t}\log(\rho K_t)\mathrm{d}t\right] \\
&= \int_0^\infty \rho e^{-\rho t}\left(\log(\rho) + \mathbb{E}\left[\log(K_t)\right]\right)\mathrm{d}t \\
&= \int_0^\infty \rho e^{-\rho t}\left(\log(\rho) + \log(K) + \left(\mu - \rho - \frac{\sigma^2+\sigma_A^2}{2}\right)t\right)\mathrm{d}t \\
&= \log(K) + \log(\rho) + \frac{\mu - \rho - \frac{\sigma^2+\sigma_A^2}{2}}{\rho} = \log(K) + \log(m_1).
\end{aligned}$$

Let $\omega \in \mathbb{R}$ be some utility level. The minimum capital level $K_{\min}(\omega)$ required to achieve ω, in the case that the principal does not consume, is given by

$$K_{\min}(\omega) = \frac{\exp(\omega)}{m_1} > 0.$$

Indeed,

$$\begin{aligned}\omega(K_0) \geq \omega &\iff \log(K_0) + \log(m_1) \geq \omega \\ &\iff \log(K_0) \geq \omega - \log(m_1) \\ &\iff K_0 \geq \frac{\exp(\omega)}{m_1}\end{aligned}$$

holds. Suppose $K_0 > 0$ and $\omega_0 \in \mathbb{R}$ are such that $\omega_0 - \log(K_0) = a_0 \geq \log(m_1)$, then $K_0 \leq K_{\min}(\omega_0)$ with equality if and only if $\omega_0 - \log(K_0) = \log(m_1)$. Hence, in the case $\omega_0 - \log(K_0) = \log(m_1)$, the principal can only guaranty the agent's utility level ω_0 by choosing zero consumption for herself. This leads to $V(K_0, \omega_0) = -\infty$ and thus $\lim_{a \to \log(m_1)} v(a) = -\infty$. If $\omega_0 - \log(K_0) > \log(m_1)$, it is not possible to achieve the utility level ω_0 for the agent with initial capital K_0.

For the limit $a \to -\infty$, we first show that $\lim_{a \to -\infty} v(a) \geq \log(m_1)$ by constructing an admissible suboptimal strategy. Define the strategy (c, c^P, β, β^A) as

$$c_t = \frac{\varepsilon}{2} K_t, \quad c_t^P = (\rho - \varepsilon) K_t, \quad \beta_t = \frac{2\rho}{\varepsilon}, \quad \beta_t^A = 1,$$

for all $t \geq 0$ and some $\varepsilon \in (0, \rho)$. Fix initial capital $K_0 > 0$. Then, the process $K = (K_t)_{t \geq 0}$ follows a geometric Brownian motion given by

$$K_t = K_0 \exp\left(\left(\mu - \rho + \frac{\varepsilon}{2} - \frac{\sigma^2 + \sigma_A^2}{2}\right) t + \sigma_A Z_t^A + \sigma Z_t\right).$$

Using this strategy, we find a lower bound for $V(K_0, \omega_0^\varepsilon)$ as

$$\begin{aligned}V(K_0, \omega_0^\varepsilon) &\geq \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \log\left((\rho - \varepsilon) K_t\right) \mathrm{d}t\right] \\ &= \log(K_0) + \log(\rho - \varepsilon) + \frac{\mu - \rho + \frac{\varepsilon}{2} - \frac{\sigma^2 + \sigma_A^2}{2}}{\rho}.\end{aligned}$$

This implies that for $a^\varepsilon = \omega_0^\varepsilon - \log(K_0)$, we have

$$v(a^\varepsilon) \geq \log(\rho - \varepsilon) + \frac{\mu - \rho + \frac{\varepsilon}{2} - \frac{\sigma^2 + \sigma_A^2}{2}}{\rho}.$$

If we let $\varepsilon \to 0$, then $a^\varepsilon \to -\infty$ since $\omega_0^\varepsilon \to -\infty$. Thus, assuming continuity of v, we find

$$\begin{aligned}\lim_{a \to -\infty} v(a) &\geq \lim_{\varepsilon \to 0} v(a^\varepsilon) \\ &\geq \lim_{\varepsilon \to 0} \left[\log(\rho - \varepsilon) + \frac{\mu - \rho + \frac{\varepsilon}{2} - \frac{\sigma^2 + \sigma_A^2}{2}}{\rho}\right] \\ &= \log(m_1).\end{aligned}$$

Let $(K_0^n, \omega_0^n)_{n\in\mathbb{N}}$ be a sequence of initial conditions such that $a^n = \omega_0^n - \log(K_0^n) \to -\infty$ as $n \to \infty$. By Proposition 2, the optimal consumptions $\hat{c}^n, \hat{c}^{P,n}$ are given by

$$\hat{c}_t^n = \frac{\rho}{y_n} K_t^n, \quad \hat{c}_t^{P,n} = \gamma_n K_t^n, \quad \forall t \geq 0,$$

for some constants $y_n, \gamma_n > 0$ satisfying

$$\gamma_n + \frac{\rho}{y_n} = \rho.$$

Moreover, the dynamics of K^n, cf. Eq. (5), is given by

$$\mathrm{d}K_t^n = \left(\mu - \gamma_n - \frac{\rho}{y_n}\right) K_t^n \mathrm{d}t + \sigma_A K_t^n \mathrm{d}Z_t^A + \sigma K_t^n \mathrm{d}Z_t.$$

Thus, K^n follows a geometric Brownian motion given by

$$K_t^n = K_0^n \exp\left(\left(\mu - \frac{\rho}{y_n} - \gamma_n - \frac{\sigma^2 + \sigma_A^2}{2}\right) t + \sigma_A Z_t^A + \sigma Z_t\right).$$

Using the optimal consumption strategy for the principal and Fubini's Theorem, we find that

$$\begin{aligned}
V(K_0^n, \omega_0^n) &= \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \log\left(\gamma_n K_t^n\right) \mathrm{d}t\right] \\
&= \int_0^\infty \rho e^{-\rho t} \left(\log\left(\gamma_n\right) + \mathbb{E}\left[\log(K_t^n)\right]\right) \mathrm{d}t \\
&= \log(K_0^n) + \log(\gamma_n) + \frac{\mu - \rho - \frac{\sigma^2 + \sigma_A^2}{2}}{\rho} \\
&\leq \log(K_0^n) + \log(\rho) + \frac{\mu - \rho - \frac{\sigma^2 + \sigma_A^2}{2}}{\rho},
\end{aligned}$$

where we use that $\gamma_n \leq \rho$, for all $n \in \mathbb{N}$. Thus, we obtain

$$v(a^n) = V(K_0^n, \omega_0^n) - \log(K_0^n) \leq \log(\rho) + \frac{\mu - \rho - \frac{\sigma^2 + \sigma_A^2}{2}}{\rho} = \log(m_1)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ and using the continuity of v, we conclude that

$$\lim_{a \to -\infty} v(a) = \lim_{n \to \infty} v(a^n) \leq \log(m_1).$$

QED

With these boundary conditions, we perform the numerical analysis using the `solve_ivp` function from the SCIPY package (Version 1.13.1).

Appendix C: Discounted infinite-horizon mean-field control problem

C.1. Mean-field limit

The purpose of this section is to describe the transition from an N-agent model to a mean-field model. In a N-agent model, the individual continuation utility of agent i will evolve as

$$d\omega_t^{i,(N)} = (\rho\omega_t^{i,(N)} - \log(c_t^i))\,dt + \sigma\frac{k_t^i}{c_t^i}dZ_t^i.$$

Hereafter, we denote by $\mathbb{P}_t^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta_{\omega_t^{i,(N)}}$ the empirical measure of continuation utilities, where δ_x denotes the Dirac measure at x. We now assume the following feedback nature of the controls:

$$c_t^i = c(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)}) \tag{85}$$

$$k_t^i = k(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)}) \tag{86}$$

$$c_t^P = c^P(K_t^{(N)}, \mathbb{P}_t^{(N)}) \tag{87}$$

for each $i = 1, ...N$, where c, c^P, and k are Lipschitz continuous[25]. The law of motion of total average capital is

$$dK_t^{(N)} = \left(\mu K_t^{(N)} - \frac{1}{N}\sum_{i=1}^N c(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)}) - c^P(K_t^{(N)}, \mathbb{P}_t^{(N)})\right)dt$$
$$+ \frac{\sigma}{N}\sum_{i=1}^N k(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)})dZ_t^i + \sigma_A K_t^{(N)} dZ_t^A.$$

From the theory of propagation of chaos (Sznitman (1991) section 1 for details), we expect that, when $N \to \infty$, for any fixed integer $m \leq N$, the $m+1$-dimensional process $(K_t^{(N)}, \omega_t^{1,(N)}, \ldots, \omega_t^{m,(N)})_{t\geq 0}$ converges in law to the vector $(K_t, \omega_t^1, \ldots, \omega_t^m)$ where

$$dK_t = \left(\mu \int k(K_t, \mathbb{P}_t, .)d\mathbb{P}_t - \int c(K_t, \mathbb{P}_t, .)d\mathbb{P}_t - c^P(K_t, \mathbb{P}_t)\right) dt + \sigma_A dZ_t^A,\ K_0 = K, \tag{88}$$

and where the ω^i are solutions to the same McKean-Vlasov SDE.

$$d\omega_t = [\rho\omega_t - \log c(K_t, \mathbb{P}_t, \omega_t)]dt + \rho\sigma y(K_t, \mathbb{P}_t, \omega_t)dZ_t + \sigma_A dZ_t^A, \tag{89}$$

where $(Z_t)_t$ is a Brownian motion and y denotes the product of functions $ku'(c)$. Conditionally on Z^A, the ω^i are independent and identically distributed with law $\mathbb{P}_t$.

[25] A function f is Lipschitz continuous on $[0, \infty) \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}$ if there exists a constant $L > 0$, such that for all $(K_0, \mathbb{P}_0, \omega_0), (K_1, \mathbb{P}_1, \omega_1) \in [0, \infty) \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}$ $|f(K_1, \mathbb{P}_1, \omega_1) - f(K_0, \mathbb{P}_0, \omega_0)| \leq L(|K_1 - K_0| + |\omega_1 - \omega_0| + W_2(\mathbb{P}_1, \mathbb{P}_0))$ holds.

Without the average capital, the convergence of $(\omega_t^{1,(N)}, \dots, \omega_t^{m,(N)})$ has been proven in Carmona and Delarue (2018), Vol 1 Chapter 10, Section 10.1.3. When capital is added, the proof of Chaintron and Diez (2022), Th. 3.1 must be adapted. We sketch a proof for the case $\sigma_A = 0$ for the sake of completeness.

In a N-agent model, the dynamics with the incentive compatibility condition are

$$d\omega_t^{i,N} = b(\omega_t^{i,N}, K_t^{(N)}, \mathbb{P}_t^{(N)})\,dt + \sigma(\omega_t^{i,N}, K_t^{(N)}, \mathbb{P}_t^{(N)})\,dZ_t^i,$$

where $b(\omega, K, \mathbb{P}) = \rho(\omega - u(c(\omega, K, \mathbb{P}))$ and $\sigma(\omega, K, \mathbb{P}) = \rho\sigma k(\omega, K, \mathbb{P}) u'(c(\omega, K, \mathbb{P}))$,

$$dK_t^{(N)} = \Big(\mu K_t^{(N)} - \int c(\omega, K_t^{(N)}, \mathbb{P}_t^{(N)})\,d\mathbb{P}_t^{(N)}(\omega) - c^P(K_t^{(N)}, \mathbb{P}_t^{(N)})\Big)\,dt$$
$$+ \frac{1}{N}\sum_{n=1}^{N} k(\omega_t^{i,N}, K_t^{(N)}, \mathbb{P}_t^{(N)})\sigma\,dZ_t^i.$$

We assume the functions b and σ are Lipschitz as well as the functions c^P and

$$\hat{c}(K, \mathbb{P}) = \int c(\omega, K, \mathbb{P})\,d\mathbb{P}(\omega).$$

We also consider the limit dynamics (the McKean-Vlasov SDE)

$$d\omega_t = b(\omega_t, K_t, \mathbb{P}_t)\,dt + \sigma(\omega_t, K_t, \mathbb{P}_t) dZ_t$$

and

$$dK_t = \big(\mu K_t - \hat{c}(K_t, \mathbb{P}_t) - c^P(K_t, \mathbb{P}_t)\big)\,dt.$$

The idea is to construct N independent processes $(\bar{\omega}_t^i)_{i=1}^N$ that follow the McKean-Vlasov SDE using the same Brownian motions as the N-particle system, i.e.

$$d\bar{\omega}_t^i = b(\bar{\omega}_t^i, K_t, \mathbb{P}_t)\,dt + \sigma(\bar{\omega}_t^i, K_t, \mathbb{P}_t) dZ_t^i.$$

We define, for a fixed $m \leq N$, the error

$$\epsilon_t = \mathbb{E}[|K_t^{(N)} - K_t|^2] + \sum_{i=1}^{m} \mathbb{E}[|\omega_t^{i,N} - \bar{\omega}_t^i|^2].$$

Applying Itô's formula to $|\omega_t^{i,N} - \bar{\omega}_t^i|^2$ and taking the expectation[26], we

[26]Under standard assumptions (Lipschitz continuity and uniformly bounded second moments), the stochastic integral is a martingale

obtain for $1 \le i \le m$,

$$\mathbb{E}[|\omega_t^{i,N} - \bar{\omega}_t^i|^2] = \mathbb{E}\int_0^t 2(\omega_s^{i,N} - \bar{\omega}_s^i)(b(\omega_s^{i,N}, \mathbb{P}_s^{(N)}) - b(\bar{\omega}_s^i, \mathbb{P}_s))ds$$
$$+ \mathbb{E}\int_0^t |\sigma(\omega_s^{i,N}, \mathbb{P}_s^{(N)}) - \sigma(\bar{\omega}_s^i, \mathbb{P}_s)|^2 ds$$
$$+ \underbrace{\mathbb{E}\int_0^t 2(\omega_s^{i,N} - \bar{\omega}_s^i)(\sigma(\omega_s^{i,N}, \mathbb{P}_s^{(N)}) - \sigma(\bar{\omega}_s^i, \mathbb{P}_s))dW_s^i}_{=0 \text{ (Martingale property)}}.$$

Using Itô's formula and the Lipschitz properties of b and σ with respect to both the states ω, K and the measure $\mathbb{P}$ (in the Wasserstein distance W_2), we derive:

$$\mathbb{E}[|\omega_t^{i,N} - \bar{\omega}_t^i|^2] \le C\int_0^t \left(\mathbb{E}[|\omega_s^{i,N} - \bar{\omega}_s^i|^2 + W_2^2(\mathbb{P}_s^{(N)}, \mathbb{P}_s)]\right) ds.$$

The term $W_2^2(\mathbb{P}_s^{(N)}, \mathbb{P}_s)$ involves the empirical measure of the N continuation utilities of the agents. Using triangular inequalities for the Wasserstein distance, we obtain

$$W_2^2(\mathbb{P}_s^{(N)}, \mathbb{P}_s) \le 2\underbrace{W_2^2(\mathbb{P}_s^{(N)}, \bar{\mathbb{P}}_s^N)}_{\text{Coupling Error}} + 2\underbrace{W_2^2(\bar{\mathbb{P}}_s^N, \mathbb{P}_s)}_{\text{Sampling Error}}$$

where $\bar{\mathbb{P}}_s^N = \frac{1}{N}\sum \delta_{\bar{\omega}_s^i}$ is the empirical measure of the independent limit utilities.
According to Fournier and Guillin (2015), the sampling error satisfies for all $s \le T$,

$$\mathbb{E}[W_2^2(\bar{\mathbb{P}}_s^N, \mathbb{P}_s)] \le \frac{C_T}{\sqrt{N}}.$$

By definition of the Wasserstein distance, the coupling error satisfies

$$W_2^2(\mathbb{P}_s^{(N)}, \bar{\mathbb{P}}_s^N) \le \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[|\omega_s^{i,N} - \bar{\omega}_s^i|^2]$$
$$= \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}[|\omega_s^{i,N} - \bar{\omega}_s^i|^2].$$

We then deduce

$$\sum_{i=1}^{m}\mathbb{E}[|\omega_t^{i,N} - \bar{\omega}_t^i|^2] \le C\left(\int_0^t \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}[|\omega_s^{i,N} - \bar{\omega}_s^i|^2]\, ds + \frac{C_T}{\sqrt{N}}\right). \tag{90}$$

We now address the term $\mathbb{E}[|K_t^{(N)} - K_t|^2]$. Using Itô's formula again, we have

$$d|K_t^{(N)} - K_t|^2 = 2(K_t^{(N)} - K_t)d(K_t^{(N)} - K_t) + d\langle K_t^{(N)} - K_t\rangle.$$

We have

$$d(K_t^{(N)} - K_t) = (\mu(K_t^{(N)} - K_t) + \hat{c}(K_t, \mathbb{P}_t) + c^P(K_t, \mathbb{P}_t) - (\hat{c}(K_t^{(N)}, \mathbb{P}_t^N) + c^P(K_t^{(N)}, \mathbb{P}_t^N))\, dt$$
$$+ \frac{1}{N} \sum_{i=1}^N k(\omega_t^{i,N}, K_t^{(N)}, \mathbb{P}_t^{(N)})\, dZ_t^i.$$

and

$$d\langle K_t^{(N)} - K_t \rangle = \frac{1}{N^2} \sum_{i=1}^N k^2(\omega_t^{i,N}, K_t^{(N)}, \mathbb{P}_t^{(N)})\, dt.$$

Denoting f the Lipschitz function $\hat{c} + c^P$, we have

$$\mathbb{E}[|K_t^{(N)} - K_t|^2] = \int_0^t 2(K_s^{(N)} - K_s)(\mu(K_s^{(N)} - K_s) + f(K_s, \mathbb{P}_s) - f(K_s^{(N)}, \mathbb{P}_s^N))\, ds$$
$$+ \frac{1}{N} \sum_{i=1}^N \underbrace{\mathbb{E}\left[\int_0^t (K_s^{(N)} - K_s) k(\omega_s^{i,N}, K_s^{(N)}, \mathbb{P}_s^{(N)})\, dZ_s^i\right]}_{=0}$$
$$+ \frac{1}{N^2} \sum_{i=1}^N \mathbb{E}\left[\int_0^t k^2(\omega_s^{i,N}, K_s^{(N)}, \mathbb{P}_s^{(N)})\, ds\right].$$

We treat the last two terms separately. The first term does not pose any real difficulty once we account for inequality

$$(K_s^{(N)} - K_s)(f(K_s, \mathbb{P}_s) - f(K_s^{(N)}, \mathbb{P}_s^N)) \leq \frac{1}{2}\left(|K_s^{(N)} - K_s|^2 + (f(K_s, \mathbb{P}_s) - f(K_s^{(N)}, \mathbb{P}_s^N))^2\right).$$

It can be treated similarly to the previous case by invoking the Lipschitz continuity of f and the W_2 triangle inequality. The third one necessitates an assumption regarding the control k; this will be verified ex post for the optimal contract. The assumption reads as follows:

Assumption 19 *There exists a function ϕ such that $k(\omega, K, \mathbb{P}) \leq C\phi(\omega)$ and $\mathbb{E}[\phi^2(\omega_t)]$ is bounded for all $t \leq T$.*

Under Assumption 19, the last term is bounded above by $\frac{C}{N}$.
Collecting all the terms, we arrive at

$$\epsilon(t) \leq C \int_0^t \left(\epsilon(s) + \frac{C}{N}\right) ds$$

By Grönwall's Lemma,

$$\epsilon(t) \leq \frac{1}{N} e^{CT}.$$

This proves that for any fixed T and $m \leq N$, the random vector

$$(K_t^{(N)}, \omega_t^{1,(N)}, \dots, \omega_t^{m,(N)})_{t \geq 0}$$

converges in law to the vector $(K_t, \omega_t^1, \dots, \omega_t^m)$.
To conclude, it remains to be verified that the optimal contract in the case of the log utility yields a model consistent with the formalism described above. In that case, $b(\omega, K, \mathbb{P})$ is linear in ω and $\sigma(\omega, K, \mathbb{P})$ is a constant. The control k is given by $k(\omega) = e^{-a} \exp(w)$. Therefore, it satisfies Assumption 19 because the process ω_t along the optimal contract is a Gaussian process that admits exponential moments of any order.

C.2. Mean-field control

In a discounted infinite horizon mean-field Control problem (also known as a McKean-Vlasov control problem), an optimizer (here the principal) seeks to maximize a functional over an infinite time horizon, where the dynamics depend not only on the state and control but also on the statistical distribution of the entire population. The shift from classical control to mean-field Control (MFC) is essentially a shift to the case in which the state variable is a probability distribution.

Thus, in mean-field Control, the principal looks at the distribution of all agents $\mathbb{P}$ and asks: what is the optimal utility from this current distribution? Therefore, the principal value function in our setting is: $V : (0, \infty) \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ and writes

$$V(K, \mathbb{P}) = \sup_X \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} u(c_t^P) dt\right]. \tag{91}$$

The space of probability measures $\mathcal{P}(\mathbb{R})$ is infinite-dimensional and does not have a vector space structure. To characterize the sensitivity of the value function V with respect to variations in the measure $\mathbb{P}$, we need a definition of a derivative for a function $F : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$. Consider a real-valued function F defined on $\mathcal{P}_2(\mathbb{R})$, which is the set of probability measures on $\mathbb{R}$ with finite second moment. We endow $\mathcal{P}_2(\mathbb{R})$ with the Wasserstein distance. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $\Pi(\mu, \nu)$ is the set of transport plans, that is, probability measures on $\mathbb{R} \times \mathbb{R}$ with respective marginals μ and ν. The Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$ is defined as the square root of

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2} |y - x|^2 \, d\gamma(x, y).$$

To apply a verification argument for the principal mean-field problem, we are interested in Itô's formula for F to describe the dynamics $t \to F(\mathbb{P}_t)$, where $\mathbb{P}_t$ is the marginal conditional probability distribution of the process $(w_t)_t$ given by Equation (89). Itô's formula for F requires differential calculus on the space of measures. We start by recalling the notions of first and second variations and L-differentiability for functions of measures relying on the convexity of $\mathcal{P}_2(\mathbb{R})$ (see Carmona and Delarue Vol 2, Chapter 4, Assumptions (A1) and (A2)) .

Definition 20 *We will say that*

- *A function F admits a first variation at $\mu \in \mathcal{P}_2(\mathbb{R})$ if there exists a real-valued and continuous function $\nabla F[\mu] : \mathbb{R} \to \mathbb{R}$, such that for all ν in $\mathcal{P}_2(\mathbb{R})$, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F((1-\varepsilon)\mu + \varepsilon\nu) - F(\mu) \right) = \int_{\mathbb{R}} \nabla F[\mu](\omega) \, d(\nu - \mu)(\omega).$$

- *A function F admits a second variation at $\mu \in \mathcal{P}_2(\mathbb{R})$ denoted $\nabla^2 F(\mu)(.)$ if for every ω, the function $\nabla F[.](\omega)$ admits a first variation at μ.*

- *A function F that admits a first variation at $\mu \in \mathcal{P}_2(\mathbb{R})$ is L-differentiable at μ if the function $\nabla F[\mu]$ is twice differentiable on $\mathbb{R}$. We will denote by $\partial^2_\omega \nabla F[\mathbb{P}]$ and $\partial_{\omega\omega} \nabla F[\mathbb{P}]$ its first and second L-derivatives.*

For a function F that is L-differentiable and in the absence of aggregate shock, Itô's formula, associated to the dynamics $t \to F(\mathbb{P}_t)$ with the dynamics of ω_t given in (89), takes the following form (see Carmona and Delarue (2018), Vol 1, Chapter 5, Th. 5.92):

$$\begin{aligned} F(\mathbb{P}_t) = F(\mathbb{P}_0) &+ \textstyle\int_0^t \mathbb{E}\left[\partial_\omega \nabla F[\mathbb{P}_s](\omega_s)(\rho\omega_s - \log c(K_s, \mathbb{P}_s, \omega_s))\right] ds \\ &+ \tfrac{1}{2} \textstyle\int_0^t \mathbb{E}\left[\partial_{\omega\omega} \nabla F[\mathbb{P}_s](\omega_s) \sigma^2 y^2(K_s, \mathbb{P}_s, \omega_s)\right] ds. \end{aligned} \tag{92}$$

In the presence of the aggregate shock, we have (see Carmona and Delarue (2018), Vol 2 chap. 4, Th 4.14)

$$\begin{aligned} dF(\mathbb{P}_t) = &\left(\int_{\mathbb{R}} \partial_\omega \nabla F(\mathbb{P}_t)(\omega)(\rho\omega - \log c(K_t, \mathbb{P}_t, \omega))\mathbb{P}_t(d\omega) \right) dt \\ &+ \frac{1}{2} \int_{\mathbb{R}} \left([\sigma^2 y^2(\omega, \mathbb{P}_t, u_t) + \sigma_A^2] \partial^2_{\omega\omega} \nabla F(\mathbb{P}_t)](\omega) \right) \mathbb{P}_t(d\omega) dt \\ &+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\sigma_A^2 \partial^2_{\omega\omega'} \nabla^2 F(\mathbb{P}_t)(\omega, \omega') \right) \mathbb{P}_t(d\omega)\mathbb{P}_t(d\omega') dt \\ &+ \left(\int_{\mathbb{R}} \partial_\omega \nabla F(\mathbb{P}_t)(\omega) \sigma_A \mathbb{P}_t(d\omega) \right) dZ_t^A \end{aligned}$$

By applying the dynamic programming principle, the Itô's formula above allows for the derivation of the HJB equation (39) for the value function V.

The class of L-differentiable functions that appears in our setting can be described as follows. Let ϕ be a twice continuously differentiable function on $\mathbb{R}$ with quadratic growth and v a continuously differentiable function on $\mathbb{R}$. We consider the function F defined on $\mathcal{P}_2(\mathbb{R})$ by

$$F(\mu) = v\left(\int_{\mathbb{R}} \phi(\omega)\mu(d\omega) \right).$$

Then, F is L-differentiable with

$$\nabla F[\mu](\omega) = v'\left(\int_{\mathbb{R}} \phi(\omega)\mu(d\omega)\right)\phi(\omega), \nabla^2 F[\mu](\omega,\omega') = v''\left(\int_{\mathbb{R}} \phi(\omega)\mu(d\omega)\right)\phi(\omega)\phi(\omega'),$$

$$\partial_\omega \nabla F[\mu] = v'\left(\int_{\mathbb{R}} \phi(\omega)\mu(d\omega)\right)\phi' \text{ and } \partial^2_{\omega\omega}\nabla F[\mu] = v'\left(\int_{\mathbb{R}} \phi(\omega)\mu(d\omega)\right)\phi''.$$

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