Dissecting Anomalies in Conditional Asset Pricing

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Abstract

We develop a methodology for estimating and testing the effect of anomalies in conditional asset pricing models when premia are time-varying. Our method, which builds on the two-pass methodology, is developed for ordinary and weighted least-squares estimation, considering both cases of correct specification and global misspecification of the candidate asset pricing model. A cross-sectional R-squared test to dissect anomalies is proposed, establishing its limiting properties under the null hypothesis of no effect of anomalies and its alternative. Using a dataset of $20,000$ individual US stock returns, we find that although anomalies are statistically significant in about half the cases (out of 170 anomalies), they explain a small fraction (less than 10%) of the cross-sectional variation of expected returns. Anomalies tend to be more important during economic and financial crises.

Keywords: Anomalies, time-variation, two-pass methodology, OLS, WLS, global misspecification, cross-sectional R-square, large-N asymptotics.

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1 Introduction

This paper provides a general methodology to formally estimate and test for the economic significance of asset pricing anomalies, within conditional asset pricing models, when both risk premia and risk exposures are allowed to be time-varying.

Asset pricing theory implies that the cross-sectional variation in expected returns should be explained by the loadings to systematic risk factors (Sharpe (1964) and Lintner (1965)). However, over decades, researchers have identified many "anomalies", where some firm- or asset-specific characteristics can predict the cross-section of expected returns, even after controlling for risk factors and their risk exposures. Examples of such anomalies include momentum (Jegadeesh and Titman (1993)), the NASDAQ anomaly (Brennan, Chordia, and Subrahmanyam (1998)), firm size and the book-to-market ratio (Fama and French (1993)), liquidity (Acharya and Pedersen (2005) and Brennan, Chordia, Subrahmanyam, and Tong (2012)), carry (Koijen, Moskowitz, Pedersen, and Vrugt (2018)), and idiosyncratic volatility (Ang, Hodrick, Xing, and Zhang (2006)), among many others. Hou, Chen, and Zhang (2020) document 452 anomalies.

Despite the very extensive literature on asset pricing anomalies, identifying and understanding such seemingly anomalous predictability represents one of the biggest challenges in empirical asset pricing theory, as it is essential for commanding what an investor considers to be risk. The availability of rigorous methodological approaches to estimate and test for anomalies is therefore of paramount importance. As highlighted by Fama and French (2008) and Hou, Chen, and Zhang (2020), the main difficulties in detecting anomalies are very often related to the availability of methodologies with which anomalies are identified. Broadly speaking, there are two main standard approaches to test for anomalies: (i) sorting average returns on anomaly variables, and (ii) using anomaly variables as additional regressors in the Fama and MacBeth (1973) two-pass regression. In this latter case, the conventional approach involves T cross-sectional ordinary least squares regressions (CSR OLS hereafter) of asset returns on the anomaly variables, one for each time period, and then interpreting the average of the T slopes' estimates as the anomaly's premia.[1]

While sorting offers an immediate picture of how returns vary across the spectrum of the anomaly variable(s), it becomes unfeasible when the sort is made on more than three variables

[1]This coincides with the second step of the two-pass Fama and MacBeth (1973) regression where, however, one omits the estimated factor loadings (i.e, the betas) from the first pass regression.

at the time and, even more importantly, it does not allow to make inference on the significance of the anomalies. On the other hand, the two-pass regression provides direct estimates of the marginal effects of each anomaly (together with their standard errors), offering a formal way to make inference on the potential existence of anomalies. However, in this paper we show that the conditions and assumptions required for its validity are very stringent and sometimes hard to justify in practice, hence invalidating many of the inferential results on the anomalies' premia.

In particular, we show that the conventional approach - based on the large-T sampling scheme - provides an accurate estimation of the average anomaly premium only if one is willing to assume orthogonality between the factor betas and the anomalies. This condition appears to be at odds with our empirical evidence, where we very often find statistically-significant non-zero correlation between the estimated betas and the anomalies. Moreover, we show that the conventional approach remains ill-suited to estimate time-varying premia, because any potential time-variation in the premia would be completely obscured by averaging the T estimates. Averaging the premia estimates over very short rolling time windows would partially resolve this problem but at the cost of invalidating the large-T asymptotic theory, which underlies all the inferential results of the conventional approach. In particular, when T is asymptotically large, we show that the classical t-ratio of the average premium is downward biased whenever one assumes that the (true) anomaly premium varies over time. In other words, we could reject the null hypothesis of a zero average premium more often than we should, unless we assume that the premium is constant for every time period (and orthogonality between the factor betas and the anomalies holds).

Introducing a new methodology that resolves all these challenges is the objective of this paper. First, given the overwhelming evidence of time-varying risk premia in the empirical asset pricing literature, we design our methodology to capture time-variation in the anomalies' premia, leaving their dynamics unspecified (i.e., nonparametrically). This is accomplished by exploiting large cross-sections of size N of asset returns while keeping their time series dimension (T) fixed and possibly very small. Given the large availability of individual securities, such setting has gained significant attention in recent years, thanks also to its flexibility to handle time variation of any (non-parametric) form, hence mitigating the risk of model misspecification and potential structural breaks in the data.[2] Moreover, by allowing the use of short (i.e., small T) unbalanced panels, our

[2] Large-T asymptotic results require fully-specified parametric assumptions to capture time-variation of loadings and risk premia, such as e g., by assuming them to be linear functions of some observed state variables (see, e.g.,

small-T approach allows one to mitigate the issue of missing data, which is a frequent, yet often overlooked, feature of company fundamentals. This problem affects the time-series availability of almost any characteristic and becomes extremely severe when one needs to analyze multiple characteristics over the same time span (see, for example, the recent contributions of Bryzgalova, Lerner, Lettau, and Pelger (2022) and Freyberger, Höppner, Neuhierl, and Weber (2021) for methodologies that tackle missing financial data).

Our methodology builds on the classical Fama and MacBeth (1973) two-pass procedure and maintains its computational ease and interpretability, despite not relying on its strict assumptions. In particular, we derive novel OLS-type estimators of both risk and anomaly time-varying premia and establish their asymptotic properties, showing how to derive closed-form standard errors to conduct correct inference on model's premia. The large-N and fixed-T setting allows us to work under very mild assumptions, which can now accommodate the more realistic case of both cross- and time-correlation between returns and anomalies, in contrast to existing methodologies.

We also extend our analysis in three main directions. The first extension introduces a new weighted least square (WLS) version of the estimator. This idea is strongly motivated by the recent literature that shows that microcaps can adversely affect the significance of anomalies. Indeed, as reported by Hou, Chen, and Zhang (2020), microcaps represent only 3.2% of the aggregate NYSE-Amex-NASDAQ market capitalization, but they account for more than 60% of the traded stocks in the market.[3] In this case, performing a simple CSR of returns on anomaly variables would make the estimates very sensitive to microcap outliers (see Hou, Chen, and Zhang (2020), Green, Hand, and Zhang (2017), and Fama and French (2008)). This impact could be mitigated by a WLS estimation, which minimizes a weighted sum of squared errors. The derivation of a WLS estimator is technically challenging, due to the potential presence of both time- and cross-sectional-correlation between weights and asset returns. This is very likely especially if the weights are defined to be equal (or proportional) to assets' market capitalization. We address these challenges and establish the limiting properties of our novel WLS estimator, providing its standard errors in closed-form.

The second extension of our analysis is about robustifying our inferential results to the case of

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Gagliardini, Ossola, and Scaillet (2016)).

[3]In Hou, Chen, and Zhang (2020), microcaps identify all stocks with a market capitalization smaller than the 20th percentile in the distribution of all the NYSE stock market equity.

global model misspecification.[4] Indeed, the significance of premia estimates can be dramatically affected by the degree of model misspecification, which could arise due to the omission of potentially relevant risk factors from the model, or because one selects the wrong (or incomplete) set of anomalies.[5] To mitigate this risk, we provide asymptotically-valid standard errors of the anomalies' premia estimates, which are robust to global model misspecification.

As a third extension, we propose a cross-sectional R-squared measure, that can be used to quantify the joint effect of anomalies on the cross-section of expected returns. Indeed, although our t-ratios can be correctly used to assess the significance of a premium estimate corresponding to the single anomaly, a cross-sectional R-squared test permits quantifying the portion of the total asset variability *jointly* explained by the anomalies. For example, one might be interested in the joint effect of anomalies belonging to the category (say, e.g., all the momentum anomalies). Specifically, we establish the limiting distribution of the proposed R-squared measure under both the null hypothesis of zero anomalies' premia and the alternative hypothesis of priced anomalies.

We present an extensive empirical application using data provided by Chen and Zimmermann (2019), from which we extract 170 anomalies at the monthly frequency (January 1986 - December 2020). We find patterns of time-variation according to which the importance of anomalies emerge often during financial crises (about 70% of the times). Although statistically the contribution of anomalies appear significant (at 5% level) for about half of cases, anomalies explain a very small fraction of the cross-sectional variation of expected returns, with only 4% of them explaining above 20%, and more than half contributing to less than 1%. In contrast, the estimated betas do not show the same pattern across time, although explain a similar fraction to anomalies of the cross-section of asset returns. The large majority of the variation in the cross-section of asset returns remains unexplained.

The paper is structured as follows. Section 2 describes the main literature, while Section 3 introduces our conditional asset pricing framework. In Section 4 we provide both analytical and numerical evidence that highlights the pitfalls of the conventional large-T method used to detect anomalies. Our methodology is formalized in Sections 5 and 6, where we present our OLS-type and

[4]By global misspecification in the context of beta-pricing models, we refer to deviations, of unspecified form, from exact pricing.

[5]See, e.g., Jagannathan and Wang (1998) for the implications of model misspecification using the two-pass methodology, valid under the large-T set up.

WLS-type estimators, respectively, with their corresponding statistical analysis. Section 7 shows how to robustify our methodology to global misspecification, while Section 8 describes our cross-sectional R-square test. The empirical application is contained in Section 9. Section 10 concludes.

2 Literature Review

The literature on asset pricing anomalies is very extensive, with a list of more than 400 papers proposing (or *dissecting*) anomalies thought to be relevant in explaining the cross-sectional variation of stock returns (see Hou, Chen, and Zhang (2020) for a detailed list). These empirical findings have spurred a growing literature that tries to summarize (or *digest*) this cross-sectional variation with new risk factors.[6] However, it seems that there are still many asset-specific characteristics that cannot be explained by any common risk factors, and that still represent the major determinants of average equity returns (Daniel and Titman (1997), Lewellen (2015), and Dong, Yan, Rapach, and Zhou (2021)).

The apparent significance of such a wide range of anomalies can be in part attributed to a lack of proper statistical methodologies. A recent example is Hou, Chen, and Zhang (2020), which cast doubts on the empirical validity of 452 anomalies proposed in asset pricing and accounting literature, showing that 65% of them fail to explain the cross-section of average stock returns, with the biggest failure (96%) being observed in the trading frictions literature. This empirical finding is even more severe if one allows for multiple testing approaches (see Harvey, Liu, and Zhu (2016)).

The two-pass methodology augmented with anomalies has been studied and extended by the literature in many directions. Important results, valid under large-T and the assumption of time-invariant premia, have been provided by Jagannathan and Wang (1998), who derived the limiting distribution of the CSR OLS estimator under the null hypothesis of no effect of anomalies. Brennan, Chordia, and Subrahmanyam (1998) propose to first net out average returns from the risk exposure to common risk factors, and then to regress these risk-adjusted average returns on observed firms' characteristics, to test for the potential effect of anomalies. Their approach has been further extended by Avramov and Chordia (2006), allowing for time variation in the factors' loadings through

[6] Prominent examples are the Fama and French (1993) and Fama and French (2015) factors, Carhart (1997) and Jegadeesh and Titman (1993) momentum factors, the liquidity factors of Pástor and Stambaugh (2003) and Acharya and Pedersen (2005), the Ang, Hodrick, Xing, and Zhang (2006) idiosyncratic risk factor, the Hou, Chen, and Zhang (2015) four q-factors, and the Stambaugh and Yuan (2017) lucky factors, among many others.

observed state variables. Chordia, Goyal, and Shanken (2015) examine the two-pass estimator in situations when N is much larger than T, and where the anomaly variables are also allowed to vary over time. However, in their work, a bootstrap procedure is proposed to derive the standard errors of the premia estimator.

Going beyond linearity imposed by the two-pass methodology, alternative approaches have been also proposed to quantify the economic relevance of anomalies. Important examples are the semi-parametric estimation of Connor and Linton (2007), the Projected Principal Component Analysis of Fan, Liao, and Wang (2016), the Instrumented Principal Component analysis of Kelly, Pruitt, and Su (2019), and the Bayesian approach of Kozak, Nagel, and Santosh (2020). Other studies have also quantified the impact of firms' characteristics on investors' portfolio choices (see, e.g., DeMiguel, Martin-Utrera, Nogales, and Uppal (2020) and Kim, Korajczyk, and Neuhierl (2021)). Using non-parametric methods, Freyberger, Neuhierl, and Weber (2020) show that characteristics play a crucial role in terms of model selection and return predictability.

Moreover, most of the empirical asset pricing literature that deals with anomalies uses portfolio data constructed from a relatively small subset of asset-specific predictors. However, although the use of portfolios reduces the sampling variability of estimated loadings, it sensibly reduces returns' heterogeneity (see Ang, Liu, and Schwarz (2020)) and makes it impossible to investigate the joint effect of a high-dimensional set of anomalies. In addition, portfolio data could be highly sensitive to data-snooping biases, especially when the same data set is repetitively examined (see Lo and MacKinlay (1990), Brennan, Chordia, and Subrahmanyam (1998), Conrad, Cooper, and Kaul (2003), Barras, Scaillet, and Wermers (2010), McLean and Pontiff (2016), and Chen (2021)). These issues can be sensibly mitigated (if not entirely avoided) using our approach, which applies to large cross-sections of individual assets, much less scrutinized than portfolio data sets.

3 Conditional Asset Pricing with Anomalies

Given our objective of estimating and testing for anomalies in a time-varying setting, the first step of our analysis requires the introduction of a *conditional* asset pricing factor model that admits the presence of anomalies. We assume that asset returns are governed by the following conditional asset pricing factor model:

$$R_{it} \quad = \quad \alpha_{i,t-1} + \boldsymbol{\beta}_i' \mathbf{f}_t + \epsilon_{it}, \quad \text{for } i = 1, \cdots, N, \ \ t = 1, \ldots, T \tag{1}$$

where R_{it} represents the gross return of stock i at time t, $\alpha_{i,t-1}$ is a potentially time-varying and asset specific intercept, $\boldsymbol{\beta}_i = (\beta_{i1}, \ldots, \beta_{iK_{\mathrm{f}}})'$ is the vector of constant loadings on K_{f} *observed* risk factors $\mathbf{f}_t = (f_{1t}, \ldots, f_{K_{\mathrm{f}}t})'$, and ϵ_{it} is the asset-specific error component.[7] Using matrix notation, the asset pricing model in (1) can be re-written as

$$\mathbf{R}_t = \boldsymbol{\alpha}_{t-1} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\epsilon}_t, \tag{2}$$

where $\mathbf{R}_t$ denotes the $N \times 1$ vector of asset returns at time t, $\boldsymbol{\alpha}_{t-1} \equiv [\alpha_{1,t-1}, \ldots, \alpha_{N,t-1}]'$, $\mathbf{B} \equiv (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N)'$, and $\boldsymbol{\epsilon}_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Nt})'$.

When conditional no-arbitrage and full diversification of the mean-variance frontier hold (see Chamberlain and Rothschild (1983, Corollary 1) and Hansen and Richard (1987) for an extension to a conditional asset pricing setup), exact pricing follows. That is:

$$\mathrm{E}\left[R_{it} | I_{t-1}, \boldsymbol{\Pi}\right] = \gamma_{0,t-1} + \boldsymbol{\gamma}_{\mathrm{f,t-1}}' \boldsymbol{\beta}_i, \tag{3}$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator, I_t represents the information set available up to time t, and $\boldsymbol{\Pi}$ defines the complete set of parameters, known to the agent when evaluating expected returns, with $\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_{\mathrm{f}}, \mathbf{B}\} \subset \boldsymbol{\Pi}$, where $\boldsymbol{\gamma}_0 = (\gamma_{0,1}, \cdots, \gamma_{0,T-1})'$ denotes the zero-beta rate vector, and $\boldsymbol{\gamma}_{\mathrm{f}} = (\boldsymbol{\gamma}_{\mathrm{f,1}}, \cdots, \boldsymbol{\gamma}_{\mathrm{f,T-1}})'$ denotes the risk premia matrix associated with the observed risk factors $\mathbf{f}_t$. However, we are specifically interested in situations where (3) might not hold and, in fact, we replace it by

$$\mathrm{E}\left[R_{it} | I_{t-1}, \boldsymbol{\Pi}\right] = a_{i,t-1} + \gamma_{0,t-1} + \boldsymbol{\gamma}_{\mathrm{f,t-1}}' \boldsymbol{\beta}_i, \tag{4}$$

for some time-varying and asset-specific *pricing errors* $\mathbf{a} \subset \boldsymbol{\Pi}$, with $\mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_{T-1})'$, and where $\mathbf{a}_{t-1} = (a_{1,t-1}, \cdots, a_{N,t-1})'$.

[7] The assumption of time-invariant loadings $\boldsymbol{\beta}_i$ is without loss of generality. In fact, our theory applies to a fixed, possibly short, time window T. Therefore, the population loadings are allowed to vary, from one time window to the other, without any constraint. Alternatively, one could assumes that time variation in the loadings is driven by some observed K_{g}-dimensional state variables $\mathbf{g}_{t-1}$, such as $\boldsymbol{\beta}_{i,t-1} = \mathbf{B}_i \mathbf{g}_{t-1}$ for a suitable matrix of coefficients $\mathbf{B}_i$. In this case, the model in (1) can be rewritten as an asset pricing model with constant loadings with respect to the $K_{\mathrm{f}} + K_{\mathrm{g}}$ risk factors $(\mathbf{f}_t \otimes \mathbf{g}_{t-1})$ (see Gagliardini, Ossola, and Scaillet (2016)).

In this paper, we assume that the pricing errors $a_{i,t-1}$ are governed by some *observed* characteristics, represented by a $K_{\rm z} \times 1$ vector of *asset-specific* and possibly time-varying variables, $\mathbf{z}_{i,t-1}$, which we refer to as *anomalies.* Formally, we assume that

$$a_{i,t-1} = \boldsymbol{\gamma}'_{\rm z,t-1}\mathbf{z}_{i,t-1}, \quad \text{for } i = 1, \cdots, N, \;\; t = 1, \ldots, T \tag{5}$$

for some vector of coefficients $\boldsymbol{\gamma}_{\rm z} \subset \boldsymbol{\Pi}$, where $\boldsymbol{\gamma}_{\rm z} = (\boldsymbol{\gamma}_{\rm z,1}, \cdots, \boldsymbol{\gamma}_{\rm z,T-1})'$ denotes the *anomalies' premia* matrix. Clealry, should all the elements of $\boldsymbol{\gamma}_{\rm z}$ be zero, then exact pricing (3) holds, and no anomaly affects the cross-section of expected returns.

Using (5), the asset pricing relationship in (4) becomes

$$\mathrm{E}\left[R_{it}|I_{t-1}, \boldsymbol{\Pi}\right] = \gamma_{0,t-1} + \boldsymbol{\gamma}'_{\rm f,t-1}\boldsymbol{\beta}_i + \boldsymbol{\gamma}'_{\rm z,t-1}\mathbf{z}_{i,t-1}. \tag{6}$$

The expression in (6) represents our new asset pricing restriction.[8] It is worth noticing that, while allowing for anomalies, condition (6) does not necessarily represent a deviation from no-arbitrage, but only from exact pricing (see Proposition OA.5 in Section OA.9 of the Online Appendix for more details).

Now, under (6), the asset pricing model in (2) generalizes to

$$\mathbf{R}_t = \gamma_{0,t-1}\mathbf{1}_N + \mathbf{Z}_{t-1}\boldsymbol{\gamma}_{\rm z,t-1} + \mathbf{B}\boldsymbol{\delta}_{\rm f,t-1} + \boldsymbol{\epsilon}_t, \tag{7}$$

where $\mathbf{1}_N$ denotes a $N \times 1$ vector of ones, $\mathbf{Z}_{t-1} = (\mathbf{z}_{1,t-1}, \cdots, \mathbf{z}_{N,t-1})'$ represents the $N \times K_{\rm z}$ matrix of anomalies at time $t-1$, and where we set

$$\boldsymbol{\delta}_{\rm f,t-1} \equiv \boldsymbol{\gamma}_{\rm f,t-1} + \mathbf{f}_t - \mathrm{E}\left[\mathbf{f}_t|I_{t-1}, \boldsymbol{\Pi}\right], \tag{8}$$

which we denominate as the vector of *ex-post* risk premia.[9] An important special case of (7) arises when the risk factors represent returns of traded portfolios, in which case one simply replaces $\gamma_{0,t-1}$ with the gross risk-free rate ($R_{{\rm f},t-1}$), and sets $\boldsymbol{\gamma}_{{\rm f},t-1} = \mathrm{E}\left[\mathbf{f}_t|I_{t-1}, \boldsymbol{\Pi}\right] - \gamma_{0,t-1}\mathbf{1}_{K_{\rm f}}$.

Whenever the vector of anomalies' premia $\boldsymbol{\gamma}_{\rm z,t-1}$ in (7) is non-zero, we say that the anomalies *affect* (or, *are priced* in) the cross-section of expected returns through (6). The objective of this

[8]Condition (6) implies that the agent has full information on the anomaly variables $\mathbf{z}_{i,t-1}$ for every stock. If one suspects that the agent's information is not complete (for example, because firm's balance sheet data is released less frequently or with delays), then the asset pricing restriction (6) generalizes to $\mathrm{E}\left[R_{i,t}|I_{t-1}, \boldsymbol{\Pi}\right] = \gamma_{0,t-1} + \boldsymbol{\gamma}'_{\rm f,t-1}\boldsymbol{\beta}_i + \boldsymbol{\gamma}'_{\rm z,t-1}\mathrm{E}\left[\mathbf{z}_{i,t-1}|I_{t-1}, \boldsymbol{\Pi}\right]$, and all our arguments continue to be valid.

[9]The notion of *ex-post* risk premia was originally coined by Shanken (1992) to denote a noisy version of the ex-ante risk premia $\boldsymbol{\gamma}_{\rm f,t-1}$ due to the unexpected factor outcomes $\mathbf{f}_t - \mathrm{E}\left[\mathbf{f}_t|I_{t-1}, \boldsymbol{\Pi}\right]$, arising whenever one considers the fixed-T case.

paper is to provide a formal methodology to estimate the anomalies' premia $\boldsymbol{\gamma}_{\mathrm{z,t-1}}$ and test for their statistical significance, using the model specification in (7).

Before introducing our results, however, some clarifications are needed to avoid possible identification issues. In our setting, time variation in the anomaly premia $\boldsymbol{\gamma}_{\mathrm{z},t-1}$ drives the time variation in the pricing errors through (4) and (5), yielding the asset pricing model in (7). However, an observationally-equivalent specification to (7) would arise if one assumes that the betas - rather than the pricing errors - were time-varying in the anomalies. For example, consider the case where, instead of (4)-(5), one assumes that

$$\mathrm{E}\left[R_{it}|I_{t-1},\mathbf{\Pi}\right] = \gamma_{0,t-1} + \boldsymbol{\gamma}'_{\mathrm{f,t-1}}\boldsymbol{\beta}_{i,t-1}, \quad \text{with} \quad \boldsymbol{\beta}_{i,t-1} = \boldsymbol{\beta}_{0,i} + \mathbf{B}_1\mathbf{z}_{i,t-1}, \tag{9}$$

for a $K_{\mathrm{f}} \times 1$ vector $\boldsymbol{\beta}_{0,i}$ and a $K_{\mathrm{f}} \times K_{\mathrm{z}}$ matrix $\mathbf{B}_1$.[10] Then, the asset pricing model in (7) is re-obtained whenever

$$\boldsymbol{\gamma}_{\mathrm{z},t-1} = \mathbf{B}_1'\boldsymbol{\gamma}_{\mathrm{f},t-1}. \tag{10}$$

Although the restriction in (10) could be, in principle, tested for - allowing one to differentiate between time-variation of the anomaly premia through the loadings (as in (9)) and time-variation through the pricing errors (as in (5)) - we prefer to simplify the analysis and build our methodology on the hypothesis of constant betas, hence allowing for time-varying anomaly premia only trough the vector of pricing errors as in (5).

4 Two-Pass Methodology for Anomalies: Conventional Approach

The most common and intuitive approach to test for the presence of anomalies is based on the estimation of model (7) by means of the two-pass Fama and MacBeth (1973) regression. It first entails obtaining the estimated matrix of loadings $\hat{\mathbf{B}}$ from (2) through time-series OLS regressions (one for each asset) of asset returns on observed risk factors $\boldsymbol{f}_t$, and then estimating the premia parameters ($\gamma_{0,t-1}, \boldsymbol{\delta}_{\mathrm{f,t-1}}$, and $\boldsymbol{\gamma}_{\mathrm{z,t-1}}$) through CSR OLS (one for each period of time), using $\hat{\mathbf{B}}$ in (7).

However, recognizing that inference would necessarily be affected by the error-in-variable (EIV) problem due to the use of $\hat{\mathbf{B}}$ in (7) (see Shanken (1992)), Fama and French (2008) advocate

[10] When time variation in the loadings $\boldsymbol{\beta}_{i,t-1}$ is driven by variables other than anomalies, no identification issue arises. See Gagliardini, Ossola, and Scaillet (2016) for a similar specification.

estimation of the anomalies' premia by simple OLS cross-sectional regressions (one for each period of time) of $\mathbf{R}_t$ on $\mathbf{Z}_{t-1}$ and an intercept, hence excluding the estimated $\mathbf{B}$ from (7), yielding the *time-varying* anomaly premium estimator

$$\tilde{\boldsymbol{\gamma}}_{\mathrm{z},t-1} \equiv (\mathbf{Z}'_{t-1}\mathbb{M}_{1_N}\mathbf{Z}_{t-1})^{-1}\mathbf{Z}'_{t-1}\mathbb{M}_{1_N}\mathbf{R}_t, \tag{11}$$

where $\mathbb{M}_{1_N} \equiv \mathbf{I}_N - \mathbf{1}_N\mathbf{1}'_N/N$ is used to de-mean the data, with $\mathbf{I}_N$ denoting an identity matrix of dimension N. This implies that $\mathbb{M}_{1_N}\mathbf{R}_t = \mathbf{R}_t - \mathbf{1}_N\bar{R}_t$, with $\bar{R}_t \equiv \sum_{i=1}^N R_{it}/N$ denoting the cross-sectional sample average of returns. Similarly, $\mathbb{M}_{1_N}\mathbf{Z}_{t-1} = \mathbf{Z}_{t-1} - \mathbf{1}_N\bar{\mathbf{Z}}'_{t-1}$, setting $\bar{\mathbf{Z}}_{t-1} \equiv \sum_{i=1}^N \mathbf{z}_{i,t-1}/N$. Fama and French (2008) justify the approach in (11) by recognizing that $\tilde{\boldsymbol{\gamma}}_{\mathrm{z},t-1}$ is equivalent to the two-pass estimator applied to (7), whenever the loadings $\mathbf{B}$ and the anomalies $\mathbf{Z}_{t-1}$ in (7) are orthogonal to each other, an assumption claimed to hold empirically. This orthogonality condition is implied when the loadings are cross-sectionally invariant.

Inference is typically carried out in terms of the *average* premium, taking the time-series average of the premia estimates $\tilde{\boldsymbol{\gamma}}_{\mathrm{z},t-1}$ in (11). This yields the conventional *average premium* estimator

$$\bar{\tilde{\boldsymbol{\gamma}}}_{\mathrm{z}} \equiv \frac{1}{T-1}\sum_{t=2}^{T}\tilde{\boldsymbol{\gamma}}_{\mathrm{z},t-1} \tag{12}$$

for which the corresponding t-ratios is evaluated. To illustrate, consider the case of univariate regressions (i.e. $K_{\mathrm{z}} = 1$). In this case, the t-ratio of the average premium associated to the z-th anomaly is simply

$$t_{\mathrm{z}} \equiv \frac{\bar{\tilde{\gamma}}_{\mathrm{z}}}{\sqrt{\tilde{\Sigma}_{\gamma_{\mathrm{z}}}/(T-1)}}, \tag{13}$$

where $\tilde{\Sigma}_{\gamma_{\mathrm{z}}}$ is the sample variance of the CSR OLS estimates $\tilde{\gamma}_{\mathrm{z},t-1}$, namely:

$$\tilde{\Sigma}_{\gamma_{\mathrm{z}}} = \frac{1}{T-1}\sum_{t=2}^{T}(\tilde{\gamma}_{\mathrm{z},t-1} - \bar{\tilde{\gamma}}_{\mathrm{z}})^2. \tag{14}$$

The t-ratio in (13) is then compared with the critical values of the standard Normal distribution, conjecturing that the inference on $\bar{\tilde{\boldsymbol{\gamma}}}_{\mathrm{z}}$ is valid as $T \to \infty$. We denote this approach as the *conventional approach.*

Given the extensive use of the conventional approach in empirical studies (see Fama and French (2008) and Hou, Chen, and Zhang (2020), among others), it seems essential to understand the inferential properties of both the time-varying estimator in (11) and the average estimator in (12), as well as their ability to capture time-variation in the (true) premia, and the potential consequences of omitting factors' loadings from the estimation of model (7).

We now show that the statistical validity of the conventional approach is not always warranted, unless extremely strict conditions are applied. In particular, we show below that the asymptotic properties of the conventional approach crucially depend on the sampling scheme under consideration, namely the relative magnitude of N and T. Moreover, regardless of the adopted sampling scheme, the conventional t-ratios are never appropriate whenever one faces a model with time-varying premia parameters, making standard inference seriously problematic. To show our results, throughout this section for simplicity we assume that $K_{\mathrm{z}} = 1$ and consider three different sampling schemes: (i) the large-T–fixed-N case, (ii) the large-N–fixed-T case, and (iii) the large-T–large-N case. Formal derivations of the following results, including the generalization to the case of $K_{\mathrm{z}} > 1$, are reported in the Online Appendix OA.6.

Let us consider first the case (i) of $T \to \infty$ with fixed N. This situation applies, for example, when one uses a panel consisting of a small number of portfolios, for which a long time-series of data is available. As N is kept fixed in this sampling scheme, it follows that no asymptotic properties can be established for the time-varying estimator $\tilde{\gamma}_{t-1,z}$ in (11). One can only assert the unbiasedness of the estimator (11), which can be established only under some regularity conditions that include, among others, the *finite-N orthogonality* condition:

$$\mathbf{Z}'_{t-1}\mathbb{M}_{1_N}\mathbf{B} = \mathbf{0}_{N\times K_{\mathrm{f}}}, \tag{15}$$

namely the (in sample) cross-sectional orthogonality between factor betas and the anomaly variable $\mathbf{Z}_{t-1}$, with $\mathbf{0}_{N\times K_{\mathrm{f}}}$ representing the zero matrix of dimension $N \times K_{\mathrm{f}}$.

Under the same sampling scheme, instead, the average premium estimator (12) satisfies:

$$\bar{\tilde{\gamma}}_{\mathrm{z}} \to_p \bar{\gamma}^0_{\mathrm{z}} \equiv \lim_{T\to\infty} \bar{\gamma}_{\mathrm{z}}, \ \text{ with } \ \bar{\gamma}_{\mathrm{z}} \equiv \frac{1}{T-1}\sum_{t=2}^{T} \gamma_{\mathrm{z},t-1} \tag{16}$$

It follows that $\bar{\tilde{\gamma}}_{\mathrm{z}}$ converges to a constant quantity, $\bar{\gamma}^0_{\mathrm{z}}$, which we refer to as the *long-run* anomaly premium. Alternatively, (16) tells us that $\bar{\tilde{\gamma}}_{\mathrm{z}}$ consistently estimates the *constant* premium γ_{z}, whenever $\gamma_{\mathrm{z},t-1} = \gamma_{\mathrm{z}}$, for every $t = 1, .., T-1$. It is important to note that the results in (16) are valid under some regularity conditions, including again the orthogonality condition in (15).

Moreover, under some further regularity conditions (See Theorem OA.4 of the Online Appendix OA.6.1), as $T \to \infty$ and N is fixed, $\bar{\tilde{\gamma}}_{\mathrm{z}}$ is also asympotically normally distributed, such that

$$\sqrt{T}\left(\bar{\tilde{\gamma}}_{\mathrm{z}} - \bar{\gamma}_{\mathrm{z}}\right) \to_d \mathcal{N}\left(0, V_N\right),$$

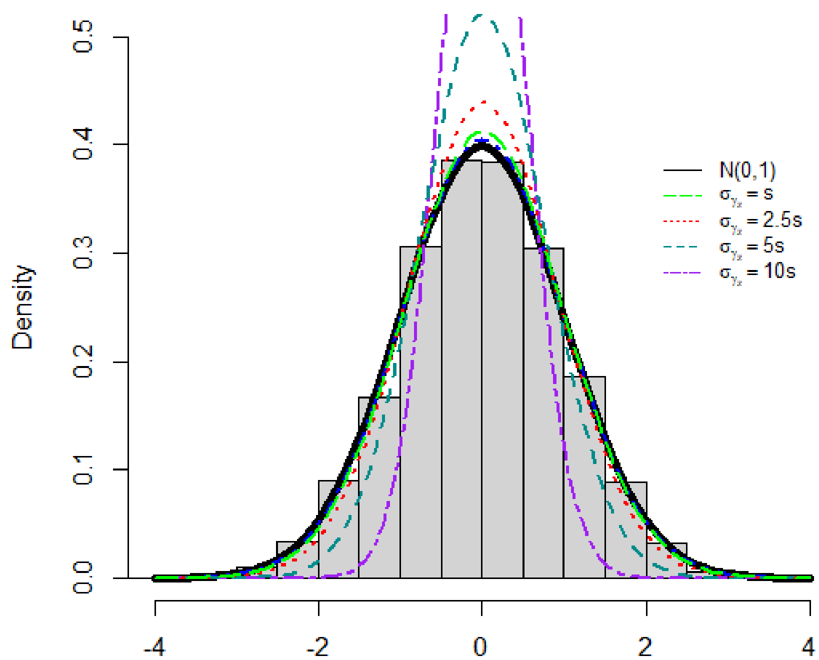
with V_N denoting the large-T asymptotic variance of the estimator, and where the subscript N is used to remark its dependency on the N-dimension as well. To conduct inference, one needs to consistently estimate V_N, which is typically done in the literature by using the the sample variance $\tilde{\Sigma}_{\gamma_{\mathrm{z}}}$ of the CSR OLS estimates, as defined in (14). However, we show that $\tilde{\Sigma}_{\gamma_{\mathrm{z}}}$ can only work in the case where the true anomaly premium is assumed to be time-invariant, i.e., when one assumes that $\gamma_{\mathrm{z},t} = \gamma_{\mathrm{z}}$ for every t in (7). More formally, we show that

$$\tilde{\Sigma}_{\gamma_{\mathrm{z}}} \to_p \sigma^2_{\gamma_{\mathrm{z}}} + V_N, \qquad \text{with} \quad \sigma^2_{\gamma_{\mathrm{z}}} \equiv \lim_{T\to\infty} \frac{1}{(T-1)} \sum_{t=2}^{T} (\gamma_{\mathrm{z},t-1} - \bar{\gamma}_{\mathrm{z}})^2. \tag{17}$$

From (17), it is immediate to see that $\tilde{\Sigma}_{\gamma_{\mathrm{z}}}$ will consistently estimate V_N only when $\sigma^2_{\gamma_{\mathrm{z}}} \equiv 0$, which happens if, and only if, $\gamma_{\mathrm{z},t-1} = \gamma_{\mathrm{z}}$ for every $t = 2, ..., T$. Whenever this condition is violated, then $\sigma^2_{\gamma_{\mathrm{z}}}$ will be a positive quantity, implying that the t-ratio in (13) is downward biased. In other words, whenever one assumes that the true premia in (7) are time-varying and uses the conventional t-ratio in (13) to make inference on the average anomaly premium, then one tends to under-reject the null hypothesis of zero (long-run) premium than prescribed by the chosen nominal size. Therefore, a statistically significant t-ratio could provide a strong indication of a non-zero average premium, even though it leaves inference undetermined when it is found to be not significant. This is an important and crucial result, which could invalidate or raise doubts on many of the findings established in the empirical literature on anomalies.

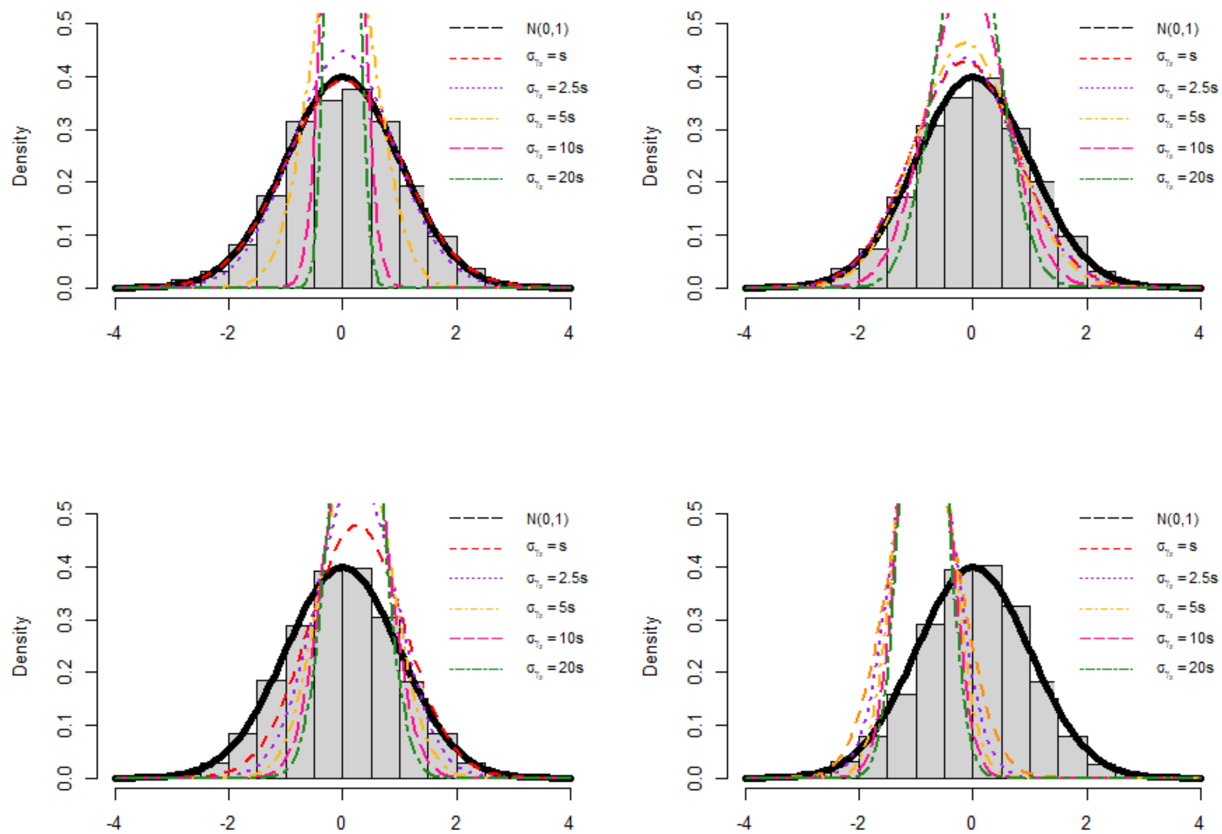
To demonstrate the potential implications of this result, we consider a simple simulation exercise, where the true anomaly premium has been generated using a time-varying scheme. Specifically, using $N = 25$ and $T = 360$, we simulate B=2,000 samples of asset returns, using the data generating process $\mathbf{R}_t = \gamma_{0,t-1}\mathbf{1}_N + \mathbf{Z}_{t-1}\gamma_{\mathrm{z,t-1}}\boldsymbol{\epsilon}_t$, where $K_{\mathrm{z}} = 1$ and $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}_N, \sigma^2_\epsilon \mathbf{I}_N)$, with $\sigma^2_\epsilon = 0.1$. For simplicity, we set $\gamma_{0,t-1} = \gamma_0 = 0$, while $\gamma_{\mathrm{z,t-1}}$ has been generated using an AR(1) model $\gamma_{\mathrm{z},t-1} = \mu_z(1-\phi_z) + \phi_z \gamma_{\mathrm{z},t-1} + u_z$, with $u_z \sim \mathcal{N}(0, \sigma^2_u)$. This implies that the variance $\sigma^2_{\gamma_{\mathrm{z}}}$ in (17) is equivalent to σ^2_u. The parameters μ_z and ϕ_z have been calibrated by fitting an AR(1) model on the estimated time series of $\tilde{\gamma}_{z,t}$, obtained by regressing observed monthly returns $\mathbf{R}_t$ on the book leverage anomaly variable $\mathbf{Z}_{t-1}$, while for σ_u^2 we consider different increasing values, from $\sigma_u^2 = s$, up to $\sigma_u^2 = 10s$, with $s = 0.001$. Then, for each simulated sample, and for each different value of σ_u^2, we estimate the anomaly average premium with (12) and construct the corresponding t-ratio in (13), which we plot in Figure 1. The figure clearly shows the inferential consequences of time-varying premia. When there is very little time variation, the classical approach works quite well (see the light green dotted curve). As the time-variation (i.e., the variance) of $\gamma_{\mathrm{z},t-1}$ increases, the distribution of the corresponding t-ratio departs substantially from the standard normal distribution, pointing to a severe under-rejection.

Figure 1: **Conventional t-ratios under a time-varying setting.** The figure shows the distribution of the conventional t-ratios in (13), when the true anomaly premium $\gamma_{\mathrm{z,t-1}}$ follows a time-varying process. Specifically, using $N = 25$ and $T = 360$, we simulate B=2,000 samples of asset returns, using the data generating process $\mathbf{R}_t = \gamma_{0,t-1}\mathbf{1}_N + \mathbf{Z}_{t-1}\gamma_{\mathrm{z,t-1}}\boldsymbol{\epsilon}_t$, where $K_{\mathrm{z}} = 1$ and $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}_N, \sigma_\epsilon^2\mathbf{I}_N)$, with $\sigma_\epsilon^2 = 0.1$. For simplicity, we set $\gamma_{0,t-1} = \gamma_0 = 0$, while $\gamma_{\mathrm{z,t-1}}$ has been generated using an AR(1) process $\gamma_{\mathrm{z},t-1} = \mu_z(1-\phi_z) + \phi_z\gamma_{\mathrm{z},t-1} + u_z$, with $u_z \sim \mathcal{N}(0, \sigma_u^2)$. This implies that the variance $\sigma_{\gamma_{\mathrm{z}}}^2$ in (17) is equivalent to $\sigma_u^2/(1-\phi_z^2)$. The parameters σ_u^2, μ_z and ϕ_z have been calibrated by fitting an AR(1) process on the estimated time series of $\tilde{\gamma}_{z,t}$, obtained by regressing observed monthly returns $\mathbf{R}_t$ on the book leverage anomaly variable $\mathbf{Z}_{t-1}$. Then, for each simulated sample, we estimate the anomaly average premium using the conventional estimator in (12) and construct the corresponding t-ratio in (13). We then plot the distribution of the B=2,000 t-ratios and repeat the same exercise for increasing values of σ_u^2. Monthly returns are from the Center for Research in Security Prices (CRSP), while data on the anomaly variables are provided by Chen and Zimmermann (2019).



The results presented above have clearly strong inferential implications which, however, provide only a partial view of the overall picture. In fact, the previous exercise assumes that the true model contains only the anomaly variables, thus excluding the estimated $\mathbf{B}$ from the return generating process. This would coincide with the two-pass estimator applied to (7), whenever the loadings $\mathbf{B}$ and the anomalies $\mathbf{Z_{t-1}}$ in (7) are orthogonal to each other. Whenever this assumption is not satisfied, the accuracy of the results could be even more compromised. The inferential consequences of excluding $\mathbf{B}$ from the estimated model are presented in Figure 2. The figure depicts the outcomes of a simulation exercise where now the true return generating process follows the model in (7), but where $\gamma_{z,t-1}$ is still estimated using (11) - hence omitting the loadings $\mathbf{B}$. Specifically, using the same parameters of the above exercise with $K_{\mathrm{f}} = 1$, we generate asset returns using the process $\mathbf{R}_t = \gamma_{0,t-1}\mathbf{1}_N + \mathbf{Z}_{t-1}\boldsymbol{\gamma}_{z,t-1} + \mathbf{B}\boldsymbol{\delta}_{\mathrm{f},t-1} + \boldsymbol{\epsilon}_{t-1}$, where $\boldsymbol{\delta}_{\mathrm{f},t-1}$ and $\mathbf{B}$ have been calibrated using data on the market factor and its loadings on observed monthly returns $\mathbf{R}_t$, respectively. To account for different degrees of correlation between $\mathbf{B}$ and $\mathbf{Z}_{t-1}$, we define the anomaly variable $\mathbf{Z}_{t-1} = [\theta\mathbb{M}_{\mathrm{B}} + (1-\theta)\mathbb{P}_{\mathrm{B}}]\tilde{\mathbf{Z}}_{t-1}$, where $\tilde{\mathbf{Z}}_{t-1}$ has been calibrated using firms' book leverage data and where we set $\mathbb{P}_{\mathrm{B}} = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-\mathbf{1}}\mathbf{B}$, and $\mathbb{M}_{\mathrm{B}} = \boldsymbol{I}_{\mathrm{N}} - \mathbb{P}_{\mathrm{B}}$. The parameter θ ranges between 0 and 1, where $\theta = 0$ represents the case of perfect correlation between $\mathbf{B}$ and $\mathbf{Z}_{t-1}$, while $\theta = 1$ indicates no correlation between the loadings and the anomaly variable. In our experiment, we consider different degrees of correlation by setting $\theta = \{1, 0.75, 0.25, 0\}$. As before, for each of the $B = 2,000$ simulated samples and for each different value of θ and σ_u^2, we estimate the average anomaly premium as in (12) and construct the corresponding t-ratios defined in (13), which we then plot in Figure 2. Each panel in Figure 2 corresponds to a different value of the parameter θ, namely $\theta = 1$ (top-left panel), $\theta = 0.75$ (top-right panel), $\theta = 0.25$ (bottom-left panel), and $\theta = 0$ (bottom-right panel). As expected, when $\theta = 1$, we re-obtain the same results of Figure 1, confirming the fact that the estimator in (11) coincides with the conventional two-pass estimator applied to (7), whenever $\mathbf{B}$ and $\mathbf{Z}_{t-1}$ are orthogonal to each other. However, as the correlation between the anomaly variable and the loadings increases, the estimation bias becomes more pronounced and combines with the downward bias arising from time variation in the anomaly premium process.

Figure 2: The figure shows the outcomes of a simulation exercise where the true return generating process follows the model in (7), but where $\gamma_{z,t-1}$ is estimated using (11) - hence omitting the loadings $\mathbf{B}$. Specifically, using the same parameters of the exercise described in Figure 1 with $K_{\mathrm{f}} = 1$, asset returns have generated using the process $\mathbf{R}_t = \gamma_{0,t-1}\mathbf{1}_{\mathrm{N}} + \mathbf{Z}_{t-1}\boldsymbol{\gamma}_{\mathrm{z,t-1}} + \mathbf{B}\boldsymbol{\delta}_{\mathrm{f},t-1} + \boldsymbol{\epsilon}_{t-1}$, where $\boldsymbol{\delta}_{\mathrm{f},t-1}$ and $\mathbf{B}$ have been calibrated using data on the market factor and its loadings on observed monthly returns $\mathbf{R}_t$, respectively. To account for different degrees of correlation between $\mathbf{B}$ and $\mathbf{Z}_{t-1}$, the anomaly variable has been generate as $\mathbf{Z}_{t-1} = [\theta\mathbb{M}_{\mathrm{B}} + (1-\theta)\mathbb{P}_{\mathrm{B}}]\tilde{\mathbf{Z}}_{\mathbf{t-1}}$, where $\tilde{\mathbf{Z}}_{\mathbf{t-1}}$ has been calibrated using firms' book leverage data and where we set $\mathbb{P}_{\mathrm{B}} = \mathbf{B}(\mathbf{B}'\mathbf{B})^{-\mathbf{1}}\mathbf{B}$, and $\mathbb{M}_{\mathrm{B}} = \mathbf{I}_{\mathrm{N}} - \mathbb{P}_{\mathrm{B}}$. The parameter θ ranges between 0 and 1, where $\theta = 0$ represents the case of perfect correlation between $\mathbf{B}$ and $\mathbf{Z}_{t-1}$, while $\theta = 1$ indicates no correlation between the loadings and the anomaly variable. The experiment considers different degrees of correlation, setting $\theta = \{1, 0.75, 0.25, 0\}$. Then, for each of the $B = 2,000$ simulated samples and for each different value of θ and σ_u^2, the average anomaly premium is estimated using (12) and the corresponding t-ratios defined in (13) have been plotted. Each panel of the figure corresponds to a different value of the parameter θ, namely $\theta = 1$ (top-left panel), $\theta = 0.75$ (top-right panel), $\theta = 0.25$ (bottom-left panel), and $\theta = 0$ (bottom-right panel). Monthly returns are from the Center for Research in Security Prices (CRSP), while data on the anomaly variables are provided by Chen and Zimmermann (2019).



Let us now consider the case of estimating (7) using the time-varying estimator in (11) and the average estimator in (12), when now $N \to \infty$ with fixed T. This situation commonly arises when one uses data on the thousands of individual stock returns - rather than portfolios - over short time windows. Under suitable regularity conditions, and assuming the *large-N orthogonality* condition

$$\frac{\mathbf{Z}'_{t-1}\mathbb{M}_{1_N}\mathbf{B}}{N} \to_p \mathbf{0}_{N\times K_{\mathrm{f}}}, \tag{18}$$

then, the time-varying estimator in (11) and the average estimator in (12) satisfy:

$$\tilde{\gamma}_{\mathrm{z},t-1} \to_p \gamma_{\mathrm{z},t-1} \quad \text{and} \quad \bar{\tilde{\gamma}}_{\mathrm{z}} \to_p \bar{\gamma}_{\mathrm{z}}. \tag{19}$$

The results in (19) imply that the time-varying estimator (11) is now able to capture the true time-varying anomaly premium, with the average estimator in (12) now converging to the *local* average premium, defined over a fixed (and possibly small) time window of length T. Moreover, when the condition in (18) is replaced by the stronger assumption in (15) - namely when the cross-sectional orthogonality condition between the factor betas and the anomaly holds in sample - we get

$$\sqrt{N}\left(\tilde{\gamma}_{\mathrm{z},t-1} - \gamma_{\mathrm{z},t-1}\right) \to_d \mathcal{N}\left(0, V_{t-1}\right), \text{ and}$$
$$\sqrt{N}\left(\bar{\tilde{\gamma}}_{\mathrm{z}} - \bar{\gamma}_{\mathrm{z}}\right) \to_d \mathcal{N}\left(0, \bar{V}\right), \quad \text{with} \quad \bar{V} = \frac{1}{(T-1)^2}\sum_{t=2}^{T} V_{t-1}$$

where V_{t-1} denotes the large-N asymptotic variance of the time-varying estimator, and where we use the subscript $t-1$ to emphasize its time dependence. However, in this large-N–fixed-T setting, inference based on conventional t-ratios becomes even more problematic than the previous large-T case, for both the time-varying and the average estimators. Indeed, the finite-T sampling scheme implies that

$$\tilde{\Sigma}_{\gamma_{\mathrm{z}}} \to_p \frac{1}{(T-1)}\sum_{t=2}^{T}(\gamma_{\mathrm{z},t-1} - \bar{\gamma}_{\mathrm{z}})^2, \tag{20}$$

which is now a positive constant that could be, in general, bigger or smaller than $\bar{V}$, making any conclusion on the over- or under-rejection of the t-ratio in (13) impossible. Moreover, notice that the conventional t-ratios would involve the incorrect $\sqrt{T}$-normalization, rather than $\sqrt{N}$, even though this would be easy to rectify. Therefore, under the large-N–fixed-T sampling scheme, except for the special circumstance when condition (18) holds, the two conventional estimators in (11) and (12) could not be used to estimate the time-varying premia in (7) and its time-average, respectively.

Moreover, a new inferential theory would be needed in this case, to equip the results with correct standard errors and t-ratios. Filling this gap is one of the objective of this paper.

Finally, let us consider the case where both N and T are allowed to diverge. Under this setting, it is easy to show that the time-varying estimator $\tilde{\gamma}_{z,t-1}$ in (11) maintains the same identical behavior of the large-N–fixed-T case discussed above, so we omit the discussion to avoid repetition. Instead, for the average estimator, we get

$$\sqrt{NT}(\bar{\tilde{\gamma}}_z - \bar{\gamma}_z) \rightarrow_d \mathcal{N}\left(0, \bar{\mathcal{V}}\right), \tag{21}$$

where $\bar{\mathcal{V}}$ denotes the large-(N,T) asymptotic variance of the average estimator, such that $(T-1)^{-1}\sum_{t=2}^{T} V_{t-1} \rightarrow_p \bar{\mathcal{V}}$. Notice that, in this case, the average estimator $\bar{\tilde{\gamma}}_z$ converges at the fast rate $O(\sqrt{NT})$ to the long-run risk premium. As for the previous case, inference remains still problematic if one uses conventional t-ratios based on $\tilde{\Sigma}_{\gamma_z}$.[11]

To summarize, our results show that the conventional approach is unable to capture and make inference on time-varying premia, whenever $T \rightarrow \infty$ and N is kept fixed. That is, in a model with time-varying anomaly premia as in (7), one can only hope to consistently estimate the (long-run) average anomaly premia $\bar{\gamma}_z$, but not the anomaly premia at each point in time $\gamma_{z,t}$. Inference is even more complicated in this setting, with the conventional t-ratio of the average premium being downward biased, hence making standard inferential results potentially invalid. Only in the special case of time-constant anomaly premia, then the conventional approach works, even though it would still require stringent assumptions.

Under the large-N–fixed-T setting, the conventional time-varying estimator in (11) could in principle be used to consistently estimate time-varying anomaly premia, even though the validity of this result requires that the stringent orthogonality condition (18) holds in the data. At any rate, conventional t-ratios (of both the time-varying and the average premium estimators) are not valid, rendering all the inferential results potentially highly misleading. The same conclusions hold if one considers the double-asymptotic setting, where both N and T jointly diverge. In this respect, our paper offers an important contribution to the literature to fill this gap.

Indeed, exploiting the large-N–fixed-T setting, we show below how it is possible to *adjust* the

[11] In this case, it is possible to show that inference could be carried out if one further assumes that $\boldsymbol{B} = \mathbf{0}_{N\times K}$, that is if none of the risk factors in the model is correlated with the test assets' returns. See Remark OA.24 in the Online Appendix OA.6.3 for formal derivations.

conventional time-varying estimators (11) and (12), and make them working under the presence of estimated betas in model (7) - hence resolving the EIV problem - and relaxing any orthogonality assumption between factor loadings and anomalies such as (18). Moreover, we provide the limiting distribution of a new time-varying estimator, showing how to derive closed-form standard errors to conduct valid inference when N becomes large. Essentially, our aim is to propose a time-varying methodology which is simple and easy to implement, and which is based on the Fama and MacBeth (1973) two-pass principle, uncovering the required adjustments to make it work.

To conclude this section, we would like to give a quick preview of some important implications of our new time-varying methodology, by analysing the performance of six categories of anomalies, namely, *Momentum*, *Value versus Growth*, *Investment*, *Profitability*, *Intangibles* and *Trading Frictions*, as in Hou, Chen, and Zhang (2020). We report the main results in Table I. Specifically, we use monthly firm-level characteristics data provided by Chen and Zimmermann (2019), from January 1986 to December 2020 and perform monthly cross-sectional regressions of each anomaly variable on monthly returns from the Center for Research in Security Prices (CRSP) using both the conventional approach and our proposed approach (which we define as "RZ Approach" in Table I), described in Section 5 below. In this latter case, and contrary to the conventional approach, cross-sectional regressions also consider the market factor in the model specification. We then group each anomaly in one of the above six categories using the classification adopted in Hou, Chen, and Zhang (2020) and report the main results, averaged across categories.[12] We repeat the same exercise for different time lengths, from $T = 12$ up to $T = 360$ months, using monthly rolling widows. Then, for each category, and for both the two approaches, in Table I we report: (i) the average percentage of times that the category has been found to be significant (Panel A), (ii) the average $|t|$-statistics to test the null hypothesis that the anomaly premium is equal to zero (Panel B), and (iii) the average anomaly premium (Panel C).

The downward bias of the conventional approach clearly emerges from Table I, especially when T is relatively small. Indeed, for all the categories, the percentage of significance obtained by using the conventional approach is always subtantially lower than the one we found with our approach. Noticeably, the result is stable across T for our RZ approach, suggesting its validity, whereas it changes sharply for the conventional approach. This is also confirmed in Panel B, where we find

[12] A complete list of the anomaly variables in each category is provided in Appendix OA.10.

that the RZ approach is almost always associated with a higher average $|t|$-ratio. Interestingly, for all the categories and regardless of the time-series length, the two approaches also show different average values of the anomaly premium (Panel C), suggesting that the correlation between the estimated betas and anomalies could be actually different from zero, rendering the conventional approach estimates biased.

Table I: Conventional Approach versus the RZ time-varying approach

	% of significance - Conventional Approach						% of significance - RZ Approach					
Panel A	$T = 12$	$T = 36$	$T = 72$	$T = 120$	$T = 240$	$T = 360$	$T = 12$	$T = 36$	$T = 72$	$T = 120$	$T = 240$	$T = 360$
Momentum	19.48	26.99	35.36	45.01	54.53	60.00	71.07	77.77	73.22	71.89	62.11	59.44
Value VS Growth	15.04	19.67	23.97	31.19	46.09	54.76	43.87	52.36	54.27	57.07	62.61	62.18
Investment	20.19	37.79	52.50	64.50	89.59	92.00	39.38	36.87	48.58	63.92	71.11	64.61
Profitability	12.98	15.11	19.81	25.92	33.07	37.00	36.21	44.16	43.76	42.20	38.44	45.58
Intangibles	11.63	17.33	25.06	31.71	40.32	56.00	28.08	33.72	37.26	41.18	37.36	33.54
Trade Frictions	11.31	15.91	19.88	25.45	36.55	51.70	37.76	35.09	39.08	44.10	47.80	46.48
	average $|t|$ - Conventional Approach						average $|t|$ - RZ Approach					
Panel B	$T = 12$	$T = 36$	$T = 72$	$T = 120$	$T = 240$	$T = 360$	$T = 12$	$T = 36$	$T = 72$	$T = 120$	$T = 240$	$T = 360$
Momentum	2.73	2.80	3.00	3.38	4.19	5.18	8.07	11.59	10.98	9.29	7.28	7.87
Value VS Growth	2.59	2.60	2.67	2.86	3.18	3.69	4.15	5.21	5.94	5.74	5.17	5.26
Investment	2.76	2.77	3.14	3.37	3.75	4.57	3.56	3.71	4.60	5.00	4.98	5.04
Profitability	2.60	2.48	2.45	2.39	2.94	3.32	4.24	4.70	4.97	4.83	3.76	3.19
Intangibles	2.69	2.64	2.79	2.82	2.98	3.38	3.67	4.40	4.90	4.89	4.61	5.76
Trade Frictions	2.93	3.03	3.37	3.19	3.05	3.43	4.64	4.52	4.53	4.10	3.81	3.82
	average premia - Conventional Approach						average premia- RZ Approach					
Panel C	$T = 12$	$T = 36$	$T = 72$	$T = 120$	$T = 240$	$T = 360$	$T = 12$	$T = 36$	$T = 72$	$T = 120$	$T = 240$	$T = 360$
Momentum	0.35	0.26	0.23	0.21	0.19	0.19	0.20	0.23	0.21	0.18	0.16	0.15
Value VS Growth	0.33	0.22	0.18	0.17	0.17	0.15	0.42	0.40	0.42	0.40	0.38	0.36
Investment	0.22	0.18	0.17	0.17	0.17	0.16	0.32	0.26	0.31	0.39	0.43	0.42
Profitability	0.30	0.19	0.15	0.13	0.12	0.12	0.58	0.52	0.50	0.45	0.44	0.38
Intangibles	0.31	0.21	0.18	0.17	0.15	0.14	0.45	0.53	0.56	0.61	0.66	0.73
Trade Frictions	0.36	0.24	0.21	0.19	0.17	0.16	0.72	0.47	0.38	0.35	0.34	0.24

5 Anomalies with Time-Varying Premia: OLS-Based Estimation

The results of the previous section show that the conventional approach is not valid whenever one postulates time variation in the (true) anomalies' premia and unless strict orthogonality conditions are satisfied. We now introduce our new results, valid when $N \to \infty$ and T remains fixed, and show how all these challenges related to the conventional approach can be resolved, by means of a new OLS *bias-adjusted* estimator of the time-varying premia $\boldsymbol{\delta}_{\mathrm{f},t-1}$ and $\boldsymbol{\gamma}_{\mathrm{z},t-1}$. All the results are established under several regularity conditions and mild assumptions that we report in Appendix A.1.

Consider again the *conditional* asset pricing model in (7), and rewrite it as

$$\mathbf{R}_t = \mathbf{Z}_{t-1}\boldsymbol{\gamma}_{z,t-1} + \mathbf{X}\boldsymbol{\Gamma}_{f,t-1} + \boldsymbol{\epsilon}_t \tag{22}$$

where $\mathbf{X} = (\mathbf{1}_N, \mathbf{B})$ and $\boldsymbol{\Gamma}_{f,t-1} = (\gamma_{0,t-1}, \boldsymbol{\delta}'_{f,t-1})'$, with $\boldsymbol{\delta}_{f,t-1}$ defined in (8). Since the matrix $\mathbf{X}$ in (22) is unknown, one needs first to estimate the loadings $\mathbf{B}$ to make the estimation of (22) feasible. The conventional two-pass approach typically advocates a simple OLS regression of $\mathbf{R}_t$ on an intercept and the observed risk factors $\mathbf{f}_t$, that is:

$$\hat{\mathbf{B}} \equiv \mathbf{R}'\mathbb{M}_{\mathbf{1}_{T-1}}\mathbf{F}(\mathbf{F}'\mathbb{M}_{\mathbf{1}_{T-1}}\mathbf{F})^{-1} = \mathbf{R}'\mathbf{P}, \tag{23}$$

where $\hat{\mathbf{B}} = (\hat{\boldsymbol{\beta}}_1, \ldots, \hat{\boldsymbol{\beta}}_N)'$, $\mathbf{F} = (\mathbf{f}_2, \cdots \mathbf{f}_T)'$, $\mathbf{R} = (\mathbf{R}_2, \cdots, \mathbf{R}_T)'$, and $\mathbf{P} \equiv \mathbb{M}_{\mathbf{1}_{T-1}}\mathbf{F}(\mathbf{F}'\mathbb{M}_{\mathbf{1}_{T-1}}\mathbf{F})^{-1}$, where we assume that $\mathbf{P}'\mathbf{P} = (\mathbf{F}'\mathbb{M}_{\mathbf{1}_{T-1}}\mathbf{F})^{-1} > 0$ for every T (see Assumption 2 in the Appendix A.1). The matrix $\mathbb{M}_{1_{T-1}} \equiv \mathbf{I}_{T-1} - \mathbf{1}_{T-1}\mathbf{1}'_{T-1}/(T-1)$ de-means the data, that is $\mathbb{M}_{1_{T-1}}\mathbf{R} = \mathbf{R} - \mathbf{1}_{T-1}\bar{\mathbf{R}}'$ and $\mathbb{M}_{1_{T-1}}\mathbf{F} = \mathbf{F} - \mathbf{1}_{T-1}\bar{\mathbf{f}}'$, setting $\bar{\mathbf{R}} \equiv \sum_{t=2}^{T}\mathbf{R}_t/(T-1)$ and $\bar{\mathbf{f}} \equiv \sum_{t=2}^{T}\mathbf{f}_t/(T-1)$.

It is clear that the estimator in (23) *excludes* the potential effect of the anomalies $\mathbf{Z}_{t-1}$, as well as the time variation of their premia. This could induce sources of bias in the estimates, further exacerbated if $\mathbf{f}_t$ and $\mathbf{Z}_{t-1}$ were potentially correlated across time, making $\hat{\mathbf{B}}$ clearly invalid. The following smoothness Assumption 1 permits to overcome these challenges, by constraining the time variation of the premia parameters, implying their (temporal) orthogonality with the risk factors $\mathbf{f}_t$. As the time-series dimension T gets small (and as long as $T > K_f + 1$), this assumption appears extremely mild, especially in terms of anomalies' premia, where the observed (time-varying) $\mathbf{z}_{i,t-1}$ could account for most of the time-variation of their overall contribution to expect returns.

Assumption 1 (*smoothness of the premia parameters*)**.** *The following hold:*

$$\mathbf{P}'\gamma_0 = \mathbf{0}_{K_f}, \qquad \mathbf{P}'\breve{\boldsymbol{\delta}}_f = \mathbf{0}_{K_f \times K_f}, \qquad \textit{and} \quad \mathbf{P}'\boldsymbol{\Delta}_z = \mathbf{0}_{K_f \times N},$$

setting the $(T-1) \times K_f$ *matrix* $\breve{\boldsymbol{\delta}}_f = (\breve{\boldsymbol{\delta}}_{f,1}, \cdots, \breve{\boldsymbol{\delta}}_{f,t-1})'$, with $\breve{\boldsymbol{\delta}}_{f,t-1} \equiv \boldsymbol{\delta}_{f,t-1} - \mathbf{f}_t = \boldsymbol{\gamma}_{f,t-1} - E(\mathbf{f}_t|I_{t-1}, \boldsymbol{\Pi})$, *and the* $(T-1) \times N$ *matrix*

$$\boldsymbol{\Delta}_z \equiv \begin{bmatrix} \gamma'_{z,1} - \gamma'_z & \mathbf{0}'_{K_z} & \cdots & \mathbf{0}'_{K_z} \\ \mathbf{0}'_{K_z} & \gamma'_{z,2} - \gamma'_z & \cdots & \mathbf{0}'_{K_z} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}'_{K_z} & \mathbf{0}'_{K_z} & \cdots & \gamma'_{z,t-1} - \gamma'_z \end{bmatrix} \begin{bmatrix} \mathbf{Z}'_1 \\ \mathbf{Z}'_1 \\ \vdots \\ \mathbf{Z}'_{T-1} \end{bmatrix},$$

for some constant $K_z \times 1$ *vector* $\boldsymbol{\gamma}_z$ *satisfying* $N^{-1}\sum_{i=1}^{N}(\mathbf{Z}'_i\mathbf{Z}_i)^{-1}\mathbf{Z}'_i\mathbf{R}_i \to_p \boldsymbol{\gamma}_z$.

When the risk factors are traded, $\breve{\boldsymbol{\delta}}_{\mathrm{f},t-1} = -\gamma_0 \mathbf{1}'_{K_{\mathrm{f}}}$ for every t, and Assumption 1 only concerns the zero-beta rate. In the special case of constant premia parameters, when both the test assets and the risk factors are expressed as excess returns, and assuming that a risk-free asset is also traded, then Assumption 1 is always satisfied.[13]

The additional source of bias, arising from the presence of $\mathbf{Z}_{t-1}$ in the first-pass, is instead dealt with by *orthogonalizing* the anomaly variables $\mathbf{Z}_{t-1}$ with respect to the observable factors $\mathbf{f}_t$, *before* running the two-step procedure. Therefore, $\mathbf{Z}_{t-1}$ can be interpreted as representing the *net* portion of the anomaly variables that affects expected returns, hence eliminating any *indirect* influence (or confounding effect) coming from the risk factors.[14]

To better understand the implications of the orthogonalization on the model's parameters and their corresponding interpretations, consider the case where the researcher postulates a model that involves a set of *initial* anomalies $\mathbf{Z}^{\dagger}_{t-1} = (\mathbf{z}^{\dagger}_{1,t-1}, \cdots, \mathbf{z}^{\dagger}_{N,t-1})'$, such that:

$$\mathbf{R}_t = \boldsymbol{\alpha}^{\dagger}_{t-1} + \mathbf{Z}^{\dagger}_{t-1}\boldsymbol{\gamma}_{\mathrm{z},t-1} + \mathbf{B}^{\dagger}\mathbf{f}_{t-1} + \mathbf{e_t} \tag{24}$$

where $\mathbf{B}^{\dagger}$ will be, in general, different from $\mathbf{B}$. Then, starting from (24), one can construct the *orthogonal* anomalies $\mathbf{Z}_{t-1}$ as the residuals from projecting $\mathbf{Z}^{\dagger}_{t-1}$ onto the unit constant and $\mathbf{f}_t$, implying a zero sample covariance between $\mathbf{z}_{i,t-1}$ and $\mathbf{f}_t$, and where we re-centre each $\mathbf{z}_{i,t-1}$ so that their sample mean coincides with the sample mean of $\mathbf{z}^{\dagger}_{\mathbf{i},\mathbf{t-1}}$, for every $i = 1, ..., N$. This leads to:

$$\mathbf{z}_{i,t-1} \equiv \mathbf{z}^{\dagger}_{i,t-1} - \hat{\boldsymbol{\Sigma}}_{\mathbf{z}^{\dagger}_i\mathbf{f}}\hat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{f}}(\mathbf{f}_t - \bar{\mathbf{f}}), \tag{25}$$

where $\hat{\boldsymbol{\Sigma}}_{\mathbf{z}^{\dagger}_i\mathbf{f}} = \widehat{\mathrm{Cov}}(\mathbf{z}^{\dagger}_{i,t-1}, \mathbf{f}') = \frac{1}{T-1}\mathbf{Z}^{\dagger\prime}_i\mathbf{F} - \bar{\mathbf{Z}}^{\dagger}_i\bar{\mathbf{f}}'$, and $\hat{\boldsymbol{\Sigma}}_{\mathbf{f}} = \widehat{\mathrm{Var}}(\mathbf{f}) = \frac{1}{T-1}\mathbf{F}'\mathbf{F} - \bar{\mathbf{f}}\bar{\mathbf{f}}'$, where $\mathbf{Z}^{\dagger}_i = (\mathbf{z}^{\dagger}_{i,1}, \cdots, \mathbf{z}^{\dagger}_{i,T-1})'$, $\bar{\mathbf{Z}}^{\dagger}_i = \mathbf{Z}^{\dagger\prime}_i\frac{\mathbf{1}_{T-1}}{T-1}$, and where we use $\widehat{\mathrm{Cov}}(\cdot)$ and $\widehat{\mathrm{Var}}(\cdot)$ to denote the sample covariance and sample variance estimators, respectively. Then, replacing (25) in (24), and

[13] One can avoid imposing the smoothness conditions of Assumption 1, and thus allowing for time-series dependence between the time-varying premia and the risk factors, but at the cost of more complicate expressions. In particular, (7) can be expressed as a panel data model with interactive-fixed effects:

$$\mathbf{R}_t = \boldsymbol{\alpha} + \mathbf{Z}_{t-1}\bar{\boldsymbol{\gamma}}_{\mathrm{z}} + \mathbf{B}\mathbf{f}_t + \mathbf{u}_t,$$

where the error term satisfies $\mathbf{u}_t = \boldsymbol{\xi}_t + \boldsymbol{\Delta}\mathbf{g}_t$ for an asset-specific error $\boldsymbol{\xi}_t$ and a vector of zero-mean latent factors $\mathbf{g}_t$ possibly correlated with the observed risk factors $\mathbf{f}_t$, with loadings $\boldsymbol{\Delta}$, and where $\bar{\boldsymbol{\gamma}}_{\mathrm{z}} = T^{-1}\sum_{t=1}^{T}\boldsymbol{\gamma}_{t-1,z}$. Assumption 1 implies orthogonality between $\mathbf{f}_t$ and $\mathbf{u}_t$, resurrecting the OLS estimator $\hat{\mathbf{B}}$. However, an alternative estimator for $\mathbf{B}$ exists that avoids Assumption 1 but leads to a more involved analysis of the CSR in the second pass. Details are available upon request.

[14] The orthogonalization between anomalies and risk factors implies that $\mathbf{Z}_{t-1}$ are no longer pre-determined. By standard arguments, this leads to a bias of order $O_p(T^{-1})$, which, however, turns out to be irrelevant in our large-N–fixed-T sampling scheme, given the fast rate at which the bias vanishes.

imposing the asset pricing restriction in (6), we get model (7), where, setting $\bar{\boldsymbol{\gamma}}_{\mathrm{z}} \equiv \boldsymbol{\Gamma}_z' \frac{\mathbf{1}_{T-1}}{T-1}$ with $\boldsymbol{\Gamma}_z = (\boldsymbol{\gamma}_{\mathrm{z},1}, \cdots, \boldsymbol{\gamma}_{\mathrm{z,T-1}})'$,

$$\boldsymbol{\beta}_i \equiv \boldsymbol{\beta}_i^{\dagger} + \hat{\boldsymbol{\Sigma}}_{\mathbf{f}}^{-1} \hat{\boldsymbol{\Sigma}}_{\mathbf{z}_i^{\dagger}\mathbf{f}}' \bar{\boldsymbol{\gamma}}_{\mathrm{z}}. \tag{26}$$

From (26), it is easy to see that, after the orthogonalization of the anomaly variables, the (transformed) $\mathbf{B}$ takes now into account not only the direct effect of the risk factors on the cross-section of expected returns, but also the indirect effect of $\mathbf{f}_t$, trough its possible dependence with $\mathbf{Z}_{t-1}^{\dagger}$.

This set-up is extremely convenient and allows us to estimate the matrix $\mathbf{B}$ by simply using (23), without now incurring in any source of bias coming from the exclusion of anomalies from the first-pass regression or due to the potential correlation between risk factors and anomalies. Therefore, the feasible version of (7) becomes

$$\mathbf{R}_t = \hat{\mathbf{X}} \boldsymbol{\Gamma}_{\mathrm{f},t-1} + \mathbf{Z}_{t-1} \boldsymbol{\gamma}_{\mathrm{z},t-1} + \boldsymbol{\eta}_t, \tag{27}$$

setting $\hat{\mathbf{X}} = (\mathbf{1}_N, \hat{\mathbf{B}})$, with $\hat{\mathbf{B}}$ defined in (23), $\boldsymbol{\eta}_t \equiv \boldsymbol{\epsilon}_t - (\hat{\mathbf{X}} - \mathbf{X}) \boldsymbol{\Gamma}_{\mathrm{f},t-1}$, and where $\mathbf{Z}_{t-1}$ satisfies (25), hence being uncorrelated with the risk factors. Running a single cross-sectional OLS regression on (27) yields the time-varying OLS estimator

$$\begin{bmatrix} \hat{\boldsymbol{\Gamma}}_{\mathrm{f},t-1} \\ \hat{\gamma}_{\mathrm{z},t-1} \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{X}}'\hat{\mathbf{X}} & \hat{\mathbf{X}}'\mathbf{Z}_{t-1} \\ \mathbf{Z}_{t-1}'\hat{\mathbf{X}} & \mathbf{Z}_{t-1}'\mathbf{Z}_{t-1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{X}}'\mathbf{R}_t \\ \mathbf{Z}_{t-1}'\mathbf{R}_t \end{bmatrix}, \tag{28}$$

where $\hat{\boldsymbol{\Gamma}}_{\mathrm{f},t-1} \equiv (\hat{\gamma}_{0,t-1}, \hat{\boldsymbol{\delta}}_{\mathrm{f},t-1}')'$. The estimator in (28) generalizes the conventional estimator $\tilde{\gamma}_{\mathrm{z},t-1}$ in (11) to the case of when both anomalies and (estimated) loadings are used as regressors in the feasible model. The two estimators coincide when $\hat{\mathbf{X}}'\mathbf{Z}_{t-1} = \mathbf{0}_{N \times K_{\mathrm{z}}}$, a condition which is, however, not warranted in general. When such orthogonality condition is violated, then $\hat{\boldsymbol{\gamma}}_{\mathrm{z},t-1}$ in (28) remains valid, but $\tilde{\gamma}_{\mathrm{z},t-1}$ in (11) becomes biased.[15]

Although (28) resolves the bias coming from the potential lack of orthogonality between the risk factors and the anomalies, unfortunately other sources of bias arise in our large-N–fixed-T set-up. The reason is that $\hat{\mathbf{B}}$ does not converge to $\mathbf{B}$ when T is fixed, making the OLS estimator in (28) biased due the EIV effect.[16]

[15] To clarify, notice that the orthogonality condition that we impose between $\mathbf{Z}_{t-1}$ and $\mathbf{f}_t$ represents a *time-series* restriction, which does not imply the *cross-sectional* restriction $\hat{\mathbf{X}}'\mathbf{Z}_{t-1} = \mathbf{0}_{N \times K_{\mathrm{z}}}$.

[16] Moreover, as the estimator (28) is evaluated at each point in time, a second source of bias arises (besides the

However, we show that such biases can be consistently estimated, leading to our new *bias-adjusted* CSR OLS estimator:

$$\begin{bmatrix}\hat{\mathbf{\Gamma}}^*_{\mathrm{f},t-1}\\ \hat{\gamma}^*_{\mathrm{z},t-1}\end{bmatrix} \equiv \begin{bmatrix}\hat{\mathbf{X}}'\hat{\mathbf{X}} - N\hat{\mathbf{\Lambda}}_1 & \hat{\mathbf{X}}'\mathbf{Z}_{t-1}\\ \mathbf{Z}'_{t-1}\hat{\mathbf{X}} & \mathbf{Z}'_{t-1}\mathbf{Z}_{t-1}\end{bmatrix}^{-1}\begin{bmatrix}\hat{\mathbf{X}}'\mathbf{R}_t - N\hat{\mathbf{\Lambda}}_{2,t-1}\\ \mathbf{Z}'_{t-1}\mathbf{R}_t\end{bmatrix}, \tag{29}$$

where $\hat{\mathbf{\Gamma}}^*_{\mathrm{f},t-1} \equiv (\hat{\gamma}^*_{0,t-1}, \hat{\boldsymbol{\delta}}^{*\prime}_{\mathrm{f},t-1})'$, and where we set

$$\hat{\mathbf{\Lambda}}_1 \equiv \begin{bmatrix}0 & \mathbf{0}'_{K_\mathrm{f}}\\ \mathbf{0}_{K_\mathrm{f}} & \hat{\sigma}^2\mathbf{P}'\mathbf{P}\end{bmatrix}, \qquad \hat{\mathbf{\Lambda}}_{2,t-1} \equiv \hat{\sigma}^2\begin{bmatrix}0\\ \mathbf{P}'\boldsymbol{\imath}_{t-1,T-1}\end{bmatrix}, \tag{30}$$

where $\boldsymbol{\imath}_{s,T-1}$ denotes the s-th row of the identity matrix $\mathbf{I}_{T-1}$, and where

$$\hat{\sigma}^2 \equiv \frac{\mathrm{tr}(\hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}})}{N(T-K-2)}, \tag{31}$$

with $\mathrm{tr}(\cdot)$ denoting the trace operator, $K = K_\mathrm{f} + K_\mathrm{z}$, and where $\hat{\boldsymbol{\epsilon}}$ represents the OLS residuals, defined as $\hat{\boldsymbol{\epsilon}}_i \equiv \mathbf{M}_{\tilde{\mathbf{D}}_i}\mathbf{R}_i$, with $\mathbb{M}_{\tilde{\mathbf{D}}_i} = \mathbf{I}_{T-1} - \tilde{\mathbf{D}}_i(\tilde{\mathbf{D}}_i'\tilde{\mathbf{D}}_i)^{-1}\tilde{\mathbf{D}}_i'$, and $\tilde{\mathbf{D}}_i \equiv (\mathbf{D}, \tilde{\mathbf{Z}}_i)$, with $\mathbf{D} \equiv (\mathbf{1}_{T-1}, \mathbf{F})$, and $\tilde{\mathbf{Z}}_i \equiv \mathbb{M}_{\mathbf{1}_{T-1}}\mathbf{Z}_i$.[17]

The following theorem establishes the limiting properties of our novel bias-adjusted estimator. Let $\mathbf{Z} \equiv (\mathbf{z}_1, ..., \mathbf{z}_N)'$ define the overall $N \times K_\mathrm{z}(T-1)$ matrix of anomalies, with $\mathbf{z}_i$ being the $K_\mathrm{z}(T-1) \times 1$ vector $\mathbf{z}_i \equiv \left(\mathrm{z}^{(1)}_{i,1}, \cdots, \mathrm{z}^{(1)}_{i,T-1}, \cdots, \mathrm{z}^{(K_\mathrm{z})}_{i,1}, \cdots, \mathrm{z}^{(K_\mathrm{z})}_{i,T-1}\right)'$, with $\mathrm{z}^{(j)}_{i,T-1}$ denoting the value of the jth anomaly for stock i at time t. Let $\mathbf{0}_a$ and $\mathbf{1}_a$ denote an $a \times 1$ vector of zeros and ones, respectively. The following $K_\mathrm{z}(T-1) \times K_\mathrm{z}$ matrices of constants

$$\mathbb{J} = \frac{1}{T-1}\begin{bmatrix}\mathbf{1}_{T-1} & \mathbf{0}_{T-1} & \dots & \mathbf{0}_{T-1}\\ \mathbf{0}_{T-1} & \mathbf{1}_{T-1} & \dots & \mathbf{0}_{T-1}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0}_{T-1} & \mathbf{0}_{T-1} & \dots & \mathbf{1}_{T-1}\end{bmatrix} = \left(\mathbf{I}_{K_\mathrm{z}} \otimes \frac{\mathbf{1}_{T-1}}{(T-1)}\right) = \frac{1}{T-1}\sum_{s=1}^{T-1}\mathbb{J}_s \tag{32}$$

with

$$\mathbb{J}_s = \begin{bmatrix}\boldsymbol{\iota}_{s,T-1} & \mathbf{0}_{T-1} & \dots & \mathbf{0}_{T-1}\\ \mathbf{0}_{T-1} & \boldsymbol{\iota}_{s,T-1} & \dots & \mathbf{0}_{T-1}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0}_{T-1} & \mathbf{0}_{T-1} & \dots & \boldsymbol{\iota}_{s,T-1}\end{bmatrix} = (\mathbf{I}_{K_\mathrm{z}} \otimes \boldsymbol{\iota}_{s,T-1}) \quad \text{for } 1 \le s \le T-1, \tag{33}$$

EIV) due to the fact that $\mathbf{P}'\boldsymbol{\imath}_{t-1,T-1}$ could be, in general, different from $\mathbf{0}_{K_\mathrm{f}}$. See Proposition OA.1 in the Online Appendix for a formal proof. In remark OA.19 we also show that the OLS estimator in (28) remains biased even when one assumes that T is large, but N is fixed. However, in this case, the bias would be a function of a random component, making the bias term impossible to be consistently estimated, unlike our large-N case.

[17] Note that, while computation of the OLS estimator $\hat{\boldsymbol{\beta}}_i$ only requires the regressors $\mathbf{D}$, the corresponding residuals must be evaluated with respect to both $\mathbf{D}_i$ and $\tilde{\mathbf{Z}}_i$, as it always happens in regressions with orthogonal independent variables.

are needed to evaluate the sample means of the anomaly variables and to select the s-th observation from the $\mathbf{Z}$ matrix, yielding $\mathbf{Z}\mathbb{J}_s = \mathbf{Z}_s$ and $\mathbf{Z}\mathbb{J} = (T-1)\sum_{s=1}^{T-1}\mathbf{Z}_s$. Finally, let $\otimes$, vec$(\cdot)$ and $\odot$ denote the Kronecker product, the vec operator, and the Hadamard product, respectively, and let $\to_p, \to_d$ denote convergence in probability and distribution, respectively.

Theorem 1 (Large-N consistency and asymptotic normality of the time varying bias-adjusted CSR OLS estimator)**.** *As $N \to \infty$, under Assumptions 1–7 (listed in the Appendix A.1), then*

(i)

$$\hat{\mathbf{\Gamma}}^*_{\mathrm{f},t-1} - \mathbf{\Gamma}_{\mathrm{f},t-1} = O_p\left(\frac{1}{\sqrt{N}}\right) \quad \textit{and} \quad \hat{\boldsymbol{\gamma}}^*_{\mathrm{z},t-1} - \boldsymbol{\gamma}_{\mathrm{z},t-1} = O_p\left(\frac{1}{\sqrt{N}}\right), \tag{34}$$

(ii)

$$\sqrt{N}\begin{bmatrix}\hat{\mathbf{\Gamma}}^*_{\mathrm{f},t-1} - \mathbf{\Gamma}_{\mathrm{f},t-1} \\ \hat{\boldsymbol{\gamma}}^*_{\mathrm{z},t-1} - \boldsymbol{\gamma}_{\mathrm{z},t-1}\end{bmatrix} \to_d \mathcal{N}\left(\mathbf{0}_{K+1}, \mathbf{L}^{-1}_{t-1}\mathbf{O}_{t-1}\mathbf{L}^{-1\prime}_{t-1}\right), \tag{35}$$

for some $\mathbf{L}_{t-1} > 0$ and $\mathbf{O}_{t-1}$ defined in (OA.35).[18]

Proof. See Appendix OA.4.

To conduct statistical inference, we need a consistent estimator of the asymptotic covariance matrix in (35), which we present in the next theorem.

Theorem 2 (Standard errors of the time varying bias-adjusted CSR OLS estimator)**.** *As $N \to \infty$, under Assumptions 1–7, and the identification condition $\kappa_4 = 0$,*

$$\hat{\mathbf{L}}^{-1}_{t-1}\,\hat{\mathbf{O}}_{t-1}\,\hat{\mathbf{L}}^{-1\prime}_{t-1} \;\; \to_p \;\; \mathbf{L}^{-1}_{t-1}\,\mathbf{O}_{t-1}\,\mathbf{L}^{-1\prime}_{t-1} \tag{36}$$

where

$$\hat{\mathbf{L}}_{t-1} \equiv \frac{1}{N}\begin{bmatrix}\hat{\mathbf{X}}'\hat{\mathbf{X}} - N\hat{\mathbf{\Lambda}}_1 & \hat{\mathbf{X}}'\mathbf{Z}_{t-1} \\ \mathbf{Z}'_{t-1}\hat{\mathbf{X}} & \mathbf{Z}'_{t-1}\mathbf{Z}_{t-1}\end{bmatrix}, \quad \textit{and} \quad \hat{\mathbf{O}}_{t-1} \equiv \begin{bmatrix}\hat{\mathbf{U}}_{t-1} & \hat{\sigma}^2\hat{\mathbf{G}}_{t-1}\hat{\mathbf{H}}'_{t-1} \\ \hat{\sigma}^2\hat{\mathbf{H}}_{t-1}\hat{\mathbf{G}}'_{t-1} & \hat{\mathbf{H}}_{t-1}\hat{\mathbf{\Sigma}}_{\mathrm{U}}\hat{\mathbf{H}}'_{t-1},\end{bmatrix} \tag{37}$$

with $\hat{\mathbf{U}}_{t-1} \equiv \hat{\sigma}^2\hat{\mathbf{Q}}'_{t-1}\hat{\mathbf{Q}}_{t-1}\left(\hat{\mathbf{\Sigma}}_{\mathrm{X}} - \hat{\mathbf{\Lambda}}_1\right) + \begin{bmatrix}0 & \mathbf{0}'_{K_\mathrm{f}} \\ \mathbf{0}_{K_\mathrm{f}} & \hat{\mathbf{V}}'_{t-1}\hat{\mathbf{U}}_\epsilon\hat{\mathbf{V}}_{t-1}\end{bmatrix}$, *setting* $\overline{\mathbb{M}}_{\tilde{\mathrm{D}}} \equiv N^{-1}\sum_{i=1}^N \mathbb{M}_{\tilde{\mathrm{D}}_i}$,

[18] To ease the exposition, the definition of $\mathbf{L}_{t-1}0$ and $\mathbf{O}_{t-1}$ has been relegated to the proof of the theorem (see (OA.35)).

$\hat{\mathbf{\Sigma}}_{\mathrm{X}} \equiv N^{-1}\hat{\mathbf{X}}'\hat{\mathbf{X}}$, $\hat{\mathbf{\Sigma}}_{\mathrm{ZB}} \equiv N^{-1}\mathbf{Z}'\hat{\mathbf{B}}$, $\hat{\boldsymbol{\mu}}_{\mathrm{z},T-1} \equiv N^{-1}\mathbf{Z}'\mathbf{1}_N$, *and* $\hat{\mathbf{\Sigma}}_{\mathrm{U}} \equiv (\hat{\sigma}^2 \boldsymbol{I}_{T-1} \otimes \mathbf{Z}'\mathbf{Z}/N)$, *with* $\hat{\mathbf{\Lambda}}_1$ *and* $\hat{\sigma}^2$ *defined in* (30) *and* (31)*, respectively, and we define the following matrices*

$$\begin{aligned}
\hat{\mathbf{Q}}_{t-1} &\equiv \boldsymbol{\imath}_{t-1,T-1} - \mathbf{P}\hat{\boldsymbol{\delta}}^{*}_{\mathrm{f},t-1}, \quad \hat{\mathbf{H}}_{t-1} \equiv \hat{\mathbf{Q}}'_{t-1} \otimes \mathbb{J}'_{t-1}, \\
\hat{\mathbf{G}}_{t-1} &\equiv \left[\hat{\mathbf{Q}}_{t-1} \otimes \hat{\boldsymbol{\mu}}_{\mathrm{z},T-1}, \hat{\mathbf{Q}}_{t-1} \otimes \hat{\mathbf{\Sigma}}_{\mathrm{ZB}}\right]', \text{ and} \\
\hat{\mathbf{V}}_{t-1} &\equiv (\hat{\mathbf{Q}}_{t-1} \otimes \mathbf{P}) - \left(\frac{\mathrm{vec}(\overline{\mathbf{M}}_{\tilde{\mathrm{D}}})}{(T-K-2)}\right)\hat{\mathbf{Q}}'_{t-1}\mathbf{P},
\end{aligned}$$

where $\hat{\mathbf{U}}_{\epsilon}$ *is obtained plugging* $\kappa_4 = 0$ *and* $\hat{\sigma}^4 = N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T-1}\hat{\epsilon}^4_{it}/3\,\mathrm{tr}\left(\overline{\mathbb{M}}^{(2)}_{\tilde{\mathrm{D}}}\right)$*, with* $\overline{\mathbb{M}}^{(2)}_{\tilde{\mathrm{D}}} \equiv \frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{M}_{\tilde{\mathrm{D}}_i} \odot \mathbb{M}_{\tilde{\mathrm{D}}_i}\right)$*, into* $\mathbf{U}_{\epsilon} = \mathbf{U}_{\epsilon}(\kappa_4, \sigma^4)$ *(see Remark 1 to Assumption 6).*

Proof. See Appendix OA.4.

The square root of the diagonal elements of $\hat{\mathbf{L}}^{-1}_{t-1}\hat{\mathbf{O}}_{t-1}\hat{\mathbf{L}}^{-1}_{t-1}$ in (37), divided by $\sqrt{N}$, represent the standard errors of the premia estimators $\hat{\mathbf{\Gamma}}^{*}_{\mathrm{f},t-1}$ and $\hat{\gamma}^{*}_{\mathrm{z},t-1}$, which can be used to construct asymptotically valid confidence intervals.

Theorems 1 and 2 show that our time-varying estimators $\hat{\mathbf{\Gamma}}^{*}_{\mathrm{f},t-1}$ and $\hat{\boldsymbol{\gamma}}^{*}_{\mathrm{z},t-1}$ accurately capture the true premia $\mathbf{\Gamma}_{\mathrm{f},t-1}$ and $\boldsymbol{\gamma}_{\mathrm{z},t-1}$ at any given point in time. However, when premia's time-variation is sufficiently smooth and not too abrupt, one could benefit from the time-series dimension of the panel and obtain more precise estimates of the (locally-averaged) premia parameters by means of rolling-windows average estimates.[19] In fact, smoothness of premia parameters (over a period of length T) can be reasonably assumed in our setting, as T can be chosen to be arbitrarily small, with the only requirement being that $T > (K_{\mathrm{f}} + 2)$.

Specifically, let $\bar{\mathbf{\Gamma}}_{\mathrm{f}} = \left(\bar{\gamma}_0, \bar{\boldsymbol{\delta}}'_{\mathrm{f}}\right)' \equiv (T-1)^{-1}\sum_{t=2}^{T}\mathbf{\Gamma}_{\mathrm{f},t-1}$ be the $(K_{\mathrm{f}}+1)$-vector of *locally-averaged* risk premia and recalling $\bar{\boldsymbol{\gamma}}_{\mathrm{z}} = \frac{1}{T-1}\sum_{t=2}^{T}\boldsymbol{\gamma}_{\mathrm{z},t-1}$. Let $\bar{\mathbf{Z}} = \frac{1}{(T-1)}\sum_{t=1}^{T-1}\mathbf{Z}_t$ be the $N \times K_{\mathrm{z}}$ matrix of anomalies' time-series averages. Then, by averaging the second-pass relationship in (27) across time, and noticing that $(T-1)^{-1}\sum_{t=1}^{T-1}\mathbf{Z}_{t-1}\boldsymbol{\gamma}_{\mathrm{z},t-1} = \bar{\mathbf{Z}}\bar{\boldsymbol{\gamma}}_{\mathrm{z}} + \widehat{\mathrm{Cov}}(\mathbf{Z}_{t-1}, \boldsymbol{\gamma}_{\mathrm{z},t-1})$, with $\widehat{\mathrm{Cov}}(\mathbf{Z}_{t-1}, \boldsymbol{\gamma}_{\mathrm{z},t-1}) \equiv (T-1)^{-1}\sum_{t=1}^{T-1}(\mathbf{Z}_{t-1} - \bar{\mathbf{Z}})(\boldsymbol{\gamma}_{\mathrm{z},t-1} - \bar{\boldsymbol{\gamma}}_{\mathrm{z}})$, one obtains:

$$\bar{\mathbf{R}} = \hat{\mathbf{X}}\bar{\mathbf{\Gamma}}_{\mathrm{f}} + \bar{\mathbf{Z}}\bar{\boldsymbol{\gamma}}_{\mathrm{z}} + \bar{\boldsymbol{\eta}}^{*},$$

[19] Rolling-window estimators can always be formally interpreted as non-parametric estimators (with a rectangular kernel) of the conditional (hence, time-varying) moments, providing an underpinning for their widespread popularity in empirical asset pricing, originated in Fama and MacBeth (1973).

where $\bar{\mathbf{R}} \equiv (T-1)^{-1}\sum_{t=2}^{T}\mathbf{R}_t$, and $\bar{\boldsymbol{\eta}}^* \equiv \bar{\boldsymbol{\eta}} + \widehat{\text{Cov}}(\mathbf{Z}_{t-1}, \boldsymbol{\gamma}_{\text{z},t-1})$, with $\bar{\boldsymbol{\eta}} \equiv (T-1)^{-1}\sum_{t=1}^{T}\boldsymbol{\eta}_t$. Therefore, following the same steps adopted for the time-varying estimator in (29), we can derive the OLS bias-adjusted estimator of the *locally-averaged* premia parameters as:

$$\begin{bmatrix}\hat{\boldsymbol{\Gamma}}_{\text{f}}^* \\ \hat{\boldsymbol{\gamma}}_{\text{z}}^*\end{bmatrix} \equiv \begin{bmatrix}\hat{\mathbf{X}}'\hat{\mathbf{X}} - N\hat{\boldsymbol{\Lambda}}_1 & \hat{\mathbf{X}}'\bar{\mathbf{Z}} \\ \bar{\mathbf{Z}}'\hat{\mathbf{X}} & \bar{\mathbf{Z}}'\bar{\mathbf{Z}}\end{bmatrix}^{-1}\begin{bmatrix}\hat{\mathbf{X}}'\bar{\mathbf{R}} \\ \bar{\mathbf{Z}}'\bar{\mathbf{R}}\end{bmatrix}, \tag{38}$$

where $\hat{\boldsymbol{\Lambda}}_1$ is defined in (30), and where $\hat{\boldsymbol{\Gamma}}_{\text{f}}^* \equiv \left(\hat{\gamma}_0^*, \hat{\boldsymbol{\delta}}_{\text{f}}^{*'}\right)'$. Notice that now, compared with the time-varying estimator in (29), the estimator in (38) is immune of the bias term related to $\boldsymbol{\Lambda}_2$. The reason is that this bias vanishes when one constructs the *locally-averaged* estimators (38), because $(T-1)^{-1}\sum_{t=2}^{T}\mathbf{P}\boldsymbol{\imath}_{t-1,T-1} = \mathbf{P}'\mathbf{1}_{T-1} = \mathbf{0}_{K_{\text{f}}}$ by construction (see Theorem OA.1 in the Online Appendix OA.5.2).

Under our assumptions, we can show that the estimator in (38) is $\sqrt{N}$-consistent and asymptotically normally distributed, following similar results to the ones established in Theorems 1 and 2.[20] This allows us to consistently estimate and make inference on the average premia $\bar{\boldsymbol{\Gamma}}_{\text{f}}$ and $\bar{\boldsymbol{\gamma}}_{\text{z}}$, with the further advantage of increasing the precision of the estimator by an order $O\left(1/\sqrt{T}\right)$ compared to the time-varying estimator in (29). Such precision gain can be substantial even when T is relatively small. Therefore, if one is willing to assume a sufficiently smooth time-variation of the true premia parameters over a short time period of length T, then our locally-averaged estimator becomes very attractive, as it can provide a very accurate measure of the average premia over that short time window.

As a final remark, it is worth noticing that our average premia estimator remains still very useful even when T is large, because it generalizes the conventional two-pass estimator in (11), without requiring any stringent orthogonality assumption. In this case, however, $(\hat{\boldsymbol{\Gamma}}_{\text{f}}^{*'}, \hat{\boldsymbol{\gamma}}_{\text{z}}^{*'})'$ will accurately estimate the *long-run* average of the time-varying premia, which clearly would not unveil any variation over time in the premia coefficients.[21]

These results motivate our large-N–fixed-T approach even further, making our estimators very appealing to deal with a setting where premia are genuinely varying over time.

[20] See Theorems OA.1 and OA.2 in the Online Appendix OA.5.2.

[21] For example, following Ang and Kristensen (2012), one could assume that $\left(\boldsymbol{\Gamma}_{\text{f},t}', \boldsymbol{\gamma}_{\text{z},t}'\right)' = (\boldsymbol{\Gamma}_{\text{f}}'(t/T), \boldsymbol{\gamma}_{\text{z}}'(t/T))'$, for some smooth functions $\boldsymbol{\Gamma}_{\text{f}}(\cdot)$ and $\boldsymbol{\gamma}_{\text{z}}(\cdot)$. Then, as T goes to infinity, $(\hat{\boldsymbol{\Gamma}}_{\text{f}}^{*'}, \hat{\boldsymbol{\gamma}}_{\text{z}}^{*'})'$ accurately estimate the *long-run* premia $\int_0^1 \begin{bmatrix}\boldsymbol{\Gamma}_{\text{f}}(s) \\ \boldsymbol{\gamma}_{\text{z}}(s)\end{bmatrix} ds$, which, although of interest (and assuming that such quantity exists finite), would completely mask any form of time-variation in the premia parameters.

6 Anomalies with Time-Varying Premia: WLS-Based Estimation

Fama and French (2008) and Hou, Chen, and Zhang (2020), among others, recognize that most of the empirical results on asset pricing anomalies can be seriously affected by the presence of micro-cap stocks. Small-cap equities typically show higher returns than large-cap stocks, but they also tend to have the largest cross-sectional dispersions both in terms of returns and anomaly variables. To mitigate this effect, Hou, Chen, and Zhang (2020) consider a Weighted Least Square (WLS) estimator of the premia parameters, with the weights being proportional to the corresponding stock's market capitalization.

Formally, let $\left(\hat{\mathbf{\Gamma}}^{(\mathrm{w})\prime}_{\mathrm{f},t-1}, \hat{\boldsymbol{\gamma}}^{(\mathrm{w})\prime}_{\mathrm{z},t-1}\right)'$ denote the WLS estimator of the $(K+1)$-vector of premia coefficients, where $\hat{\mathbf{\Gamma}}^{(\mathrm{w})}_{\mathrm{f},t-1} \equiv \left(\hat{\gamma}^{(\mathrm{w})\prime}_{0,t-1}, \hat{\boldsymbol{\delta}}^{(\mathrm{w})\prime}_{\mathrm{f},t-1}\right)'$ denotes the premia estimator of the zero-beta rate and the K_{f} risk factors, while $\hat{\boldsymbol{\gamma}}^{(\mathrm{w})}_{\mathrm{z},t-1}$ refers to the premia of the K_{z} anomaly variables. Let $\mathbf{W}_{t-1}$ be an $N \times N$ diagonal matrix containing the asset-specific weights at time $t-1$, i.e. $\mathbf{W}_{t-1} \equiv \mathrm{diag}\left(\mathrm{w}_{1,t-1}, ..., \mathrm{w}_{N,t-1}\right)$, where we assume $\mathrm{w}_{i,t} > 0$ for every asset i and period t without great loss of generality. Then, following Hou, Chen, and Zhang (2020), we have:

$$\begin{bmatrix} \hat{\mathbf{\Gamma}}^{(\mathrm{w})}_{\mathrm{f},t-1} \\ \hat{\boldsymbol{\gamma}}^{(\mathrm{w})}_{\mathrm{z},t-1} \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{X}}'\mathbf{W}_{t-1}\hat{\mathbf{X}} & \hat{\mathbf{X}}'\mathbf{W}_{t-1}\mathbf{Z}_{t-1} \\ \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\hat{\mathbf{X}} & \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\mathbf{Z}_{t-1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{X}}'\mathbf{W}_{t-1}\mathbf{R}_t \\ \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\mathbf{R}_t \end{bmatrix} \tag{39}$$

where, for each stock i, the weight $\mathrm{w}_{i,t-1}$ in $\mathbf{W}_{t-1}$ is given by the corresponding stock market capitalization at time $t-1$.

Similarly to the conventional time-varying OLS estimator defined in (28), we can show that analogous conclusions apply to the WLS estimator in (39). Indeed, whenever one wants to estimate time-varying premia under the traditional large-T–fixed-N setting, we show that the estimator in (39) would be invalid, because it is affected by a random (hence, unpredictable) bias.[22] In the large-N–fixed-T set-up, instead, the WLS estimator in (39) is still contaminated by several sources of bias which, however, can be consistently estimated, yielding our novel bias-adjusted CSR WLS estimator:[23]

[22] We show this result in the Online Appendix OA.5.1 - see Remark OA.19.
[23] See the Online Appendix OA.5.1 - Proposition OA.2.

$$\begin{bmatrix} \hat{\mathbf{\Gamma}}^{*(\mathrm{w})}_{\mathrm{f},t-1} \\ \hat{\boldsymbol{\gamma}}^{*(\mathrm{w})}_{\mathrm{z},t-1} \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{X}}'\mathbf{W}_{t-1}\hat{\mathbf{X}} - N\hat{\mathbf{\Lambda}}^{(\mathrm{w})}_{1,t-1} & \hat{\mathbf{X}}'\mathbf{W}_{t-1}\mathbf{Z}_{t-1} \\ \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\hat{\mathbf{X}} & \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\mathbf{Z}_{t-1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{X}}'\mathbf{W}_{t-1}\mathbf{R}_t - N\hat{\mathbf{\Lambda}}^{(\mathrm{w})}_{2,t-1} \\ \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\mathbf{R}_t \end{bmatrix}, \tag{40}$$

where $\hat{\mathbf{\Gamma}}^{*(\mathrm{w})}_{t-1} \equiv \left(\hat{\gamma}^{*(\mathrm{w})}_{0,t-1}, \hat{\boldsymbol{\delta}}^{*(\mathrm{w})'}_{\mathrm{f},t-1}\right)'$, and where we set

$$\hat{\mathbf{\Lambda}}^{(\mathrm{w})}_{1,t-1} \equiv \begin{bmatrix} 0 & \mathbf{0}'_{K_f} \\ \mathbf{0}_{K_f} & \hat{\sigma}^{2(\mathrm{w})}_{t-1}\mathbf{P}'\mathbf{P} \end{bmatrix}, \qquad \hat{\mathbf{\Lambda}}^{(\mathrm{w})}_{2,t-1} \equiv \hat{\sigma}^{2(\mathrm{w})}_{t-1} \begin{bmatrix} 0 \\ \mathbf{P}'\boldsymbol{\imath}_{t-1,T-1} \end{bmatrix} \tag{41}$$

with

$$\hat{\sigma}^{2(\mathrm{w})}_{t-1} \equiv \frac{\mathrm{tr}(\hat{\boldsymbol{\epsilon}}\mathbf{W}_{t-1}\hat{\boldsymbol{\epsilon}}')}{N(T-K-2)} \tag{42}$$

Before establishing the asymptotic properties of the WLS estimator in (40), it is important to highlight some necessary remarks. The choice of using stock's market capitalization in the $\mathbf{W}_{t-1}$ matrix makes the weighting scheme parameter-free. On one hand, this simplifies the WLS analysis, where the weights are instead typically defined as functions of unknown parameters (to be estimated) or set to be inversely proportional to the regression-error variance. On the other hand, however, market capitalization could be very likely correlated - both cross-sectionally and over time - with returns and other anomalies, making the asymptotic analysis of the estimator non-trivial. For this reason, we need to impose some conditions on the sample moments of anomalies, weights and asset-specific errors. Specifically, we assume that each asset-specific error is uncorrelated with past values of both anomaly variables and weights, but could be potentially correlated with their contemporary and future values (see Assumption 11 in Appendix A.1.1).

Moreover, the behavior of the weights plays a crucial role in determining the statistical properties of the WLS estimator, especially when $N \to \infty$. In particular, a condition that the weights should satisfy is the so-called *granuarity* assumption, which guarantees that the weights dissipate to zero sufficiently fast for every asset, as $N \to \infty$. When the granularity assumption fails, then the WLS estimator exhibits a random limit, making both estimation and inference invalid.Therefore, in the following theorems, we establish the limiting properties of the WLS estimator in (40) under the assumption that granularity holds (see Assumption 8 in Appendix A.1.1).

Theorem 3. *As $N \to \infty$, and under Assumptions 1–11,*

(i)

$$\hat{\boldsymbol{\Gamma}}^{*(\mathrm{w})}_{\mathrm{f},t-1} - \boldsymbol{\Gamma}_{\mathrm{f},t-1} = O_p\left(\frac{1}{\sqrt{N}}\right), \qquad \hat{\boldsymbol{\gamma}}^{*(\mathrm{w})}_{\mathrm{z},t-1} - \boldsymbol{\gamma}_{\mathrm{z},t-1} = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{43}$$

(ii)

$$\sqrt{N}\begin{bmatrix}\hat{\boldsymbol{\Gamma}}^{*(\mathrm{w})}_{\mathrm{f},t-1} - \boldsymbol{\Gamma}_{\mathrm{f},t-1} \\ \hat{\boldsymbol{\gamma}}^{*(\mathrm{w})}_{\mathrm{z},t-1} - \boldsymbol{\gamma}_{\mathrm{z},t-1}\end{bmatrix} \rightarrow_d \mathcal{N}\left(\mathbf{0}_{\mathrm{K}+1}, \mathbf{L}^{-1}_{t-1}\mathbf{O}^{(\mathrm{w})}_{t-1}\mathbf{L}^{-1}_{t-1}\right), \tag{44}$$

where $\mathbf{L}_{t-1}$ *is the same as in Theorem 1, and for for some* $\mathbf{O}^{(\mathrm{w})}_{t-1}$ *defined in* (OA.45).[24]

Proof. See Appendix OA.4.

The next theorem shows how to construct asymptotically-valid standard errors for the WLS estimator.[25]

Theorem 4. *As* $N \to \infty$*, under Assumptions 1–11, and the identification condition* $\kappa_4 = 0$*,*

$$\hat{\mathbf{L}}^{(\mathrm{w})-1}_{t-1}\,\hat{\mathbf{O}}^{(\mathrm{w})}_{t-1}\,\hat{\mathbf{L}}^{(\mathrm{w})-1\prime}_{t-1} \quad \rightarrow_p \quad \mathbf{L}^{-1}_{t-1}\,\mathbf{O}^{(\mathrm{w})}_{t-1}\,\mathbf{L}^{-1\prime}_{t-1} \tag{45}$$

where

$$\hat{\mathbf{L}}^{(\mathrm{w})}_{t-1} \equiv \frac{1}{N}\begin{bmatrix}\hat{\mathbf{X}}'\hat{\mathbf{X}} - N\hat{\boldsymbol{\Lambda}}^{(\mathrm{w})}_{1,t-1} & \hat{\mathbf{X}}'\mathbf{W}_{t-1}\mathbf{Z}_{t-1} \\ \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\hat{\mathbf{X}} & \mathbf{Z}'_{t-1}\mathbf{W}_{t-1}\mathbf{Z}_{t-1}\end{bmatrix}, \;\; \textit{and} \;\; \hat{\mathbf{O}}^{(\mathrm{w})}_{t-1} \equiv \hat{\lambda}_{t-1}\begin{bmatrix}\hat{\boldsymbol{\mu}}_{\mathrm{x}}\hat{\boldsymbol{\mu}}'_{\mathrm{x}} & \hat{\boldsymbol{\mu}}_{\mathrm{x}}\hat{\boldsymbol{\mu}}'_{\mathrm{z}} \\ \hat{\boldsymbol{\mu}}_{\mathrm{z}}\hat{\boldsymbol{\mu}}'_{\mathrm{x}} & \hat{\boldsymbol{\mu}}_{\mathrm{z}}\hat{\boldsymbol{\mu}}'_{\mathrm{z}}\end{bmatrix} + \hat{\mathbf{M}}^{(\mathrm{w})}_{t-1} \tag{46}$$

with

$$\hat{\mathbf{M}}^{(\mathrm{w})}_{t-1} \equiv \begin{bmatrix} 0 & \mathbf{0}'_{K_{\mathrm{f}}} & \mathbf{0}'_{K_{\mathrm{z}}} \\ \mathbf{0}_{K_{\mathrm{f}}} & \hat{\mu}^2_{\mathrm{w},t-1}\,\hat{\mathbf{V}}^{(\mathrm{w})\prime}_{t-1}\hat{\mathbf{U}}_{\epsilon}\hat{\mathbf{V}}^{(\mathrm{w})}_{t-1} & \mathbf{0}_{K_{\mathrm{f}}\times K_{\mathrm{z}}} \\ \mathbf{0}_{K_{\mathrm{z}}} & \mathbf{0}_{K_{\mathrm{z}}\times K_{\mathrm{f}}} & \hat{\mathbf{H}}^{(\mathrm{w})}_{t-1}\hat{\boldsymbol{\Sigma}}^{(\mathrm{w})}_{\mathrm{U}}\hat{\mathbf{H}}^{(\mathrm{w})\prime}_{t-1} + \hat{\mathbf{S}}^{(\mathrm{w})}_{t-1} + \hat{\mathbf{S}}^{(\mathrm{w})}_{t-1}\end{bmatrix} \tag{47}$$

setting $\hat{\boldsymbol{\mu}}_{\mathrm{x}} \equiv (1, \hat{\boldsymbol{\mu}}'_{\beta})'$*,* $\hat{\boldsymbol{\mu}}_{\beta} \equiv N^{-1}\hat{\mathbf{B}}\mathbf{1}_N$*,* $\hat{\boldsymbol{\mu}}_{\mathrm{z}} \equiv N^{-1}\mathbb{J}'\mathbf{Z}'\mathbf{1}_N$*,* $\hat{\mu}^2_{\mathrm{w},t-1} \equiv N^{-1}\mathbf{1}'_N\mathbf{W}^2_{t-1}\mathbf{1}_N$*,* $\hat{\boldsymbol{\Sigma}}^{(\mathrm{w})}_{\mathrm{U}} \equiv (\hat{\sigma}^{2(\mathrm{w})}_{t-1}\mathbf{I}_{T-1} \otimes N^{-1}\mathbf{Z}'\mathbf{Z})$*,* $\hat{\boldsymbol{\Sigma}}_{\mathrm{ZW}} \equiv N^{-1}\mathbf{Z}'\mathbf{W}$*, and* $\hat{\boldsymbol{\Sigma}}_{\mathrm{V}} \equiv \left(\hat{\sigma}^{2(\mathrm{w})}_{t-1}\mathbf{I}_{T-1} \otimes N^{-1}\sum_{i=1}^{N}\mathbf{w}_i\mathbf{w}'_i\right)$*, where* $\mathbf{w}_i \equiv (\mathrm{w}_{i,1}, ..., \mathrm{w}_{i,T-1})'$*, with* $\hat{\boldsymbol{\Lambda}}^{(\mathrm{w})}_{1,t-1}$ *and* $\hat{\sigma}^{2(\mathrm{w})}_{t-1}$ *defined in* (41) *and* (42)*, respectively, and we define the*

[24]To ease the exposition, the precise definition of $\mathbf{O}^{(\mathrm{w})}_{t-1}$ has been relegated to the proof of the theorem (see (OA.45)).

[25]We report the results without the proof, as if follows closely the proof of Theorem 2.

following matrices

$$
\begin{aligned}
\hat{\mathbf{Q}}_{t-1}^{(\mathrm{w})} &\equiv \boldsymbol{\imath}_{t-1,T-1} - \mathbf{P}\hat{\boldsymbol{\delta}}_{\mathrm{f},t-1}^{*(\mathrm{w})}, \quad \hat{\mathbf{H}}_{t-1}^{(\mathrm{w})} \equiv \hat{\mathbf{Q}}_{t-1}^{(\mathrm{w})'} \otimes \mathbb{J}_{t-1}', \\
\hat{\mathbf{Y}}_{t-1} &\equiv \hat{\mathbf{Q}}_{t-1}^{(\mathrm{w})} \otimes \boldsymbol{\imath}_{t-1,T-1}, \quad \hat{\lambda}_{t-1} \equiv \hat{\mathbf{Y}}_{t-1}'\hat{\boldsymbol{\Sigma}}_{\mathrm{V}}\hat{\mathbf{Y}}_{t-1}, \\
\hat{\mathbf{S}}_{t-1} &\equiv \hat{\boldsymbol{\mu}}_{\mathrm{z}}\hat{\mathbf{Y}}_{t-1}'(\hat{\sigma}_{t-1}^{2(\mathrm{w})}\boldsymbol{I}_{T-1} \otimes \hat{\boldsymbol{\Sigma}}_{\mathrm{ZW}}')\hat{\mathbf{H}}_{t-1}^{(\mathrm{w})'}, \\
\hat{\mathbf{V}}_{t-1}^{(\mathrm{w})} &\equiv (\hat{\mathbf{Q}}_{t-1}^{(\mathrm{w})} \otimes \mathbf{P}) - \left(\frac{\mathrm{vec}(\overline{\mathbf{M}}_{\tilde{\mathrm{D}}})}{T-K-2}\right)\hat{\mathbf{Q}}_{t-1}^{(\mathrm{w})'}\mathbf{P},
\end{aligned}
$$

where $\hat{\mathbf{U}}_\epsilon$ *is defined in Theorem 2.*

7 Anomalies with Time-Varying Premia: Global Misspecification

All the results so far established assume that the asset pricing model (7) is *correctly specified*, meaning that the true model does not omit any relevant variable (either a risk factor or an anomaly variable) or, alternatively, it does not include any irrelevant one.[26] When this assumption is violated - a very likely scenario - then the issue of *global misspecification* arises which, if ignored, could seriously compromise our inferential results.[27] Indeed, misspecification affects the standard errors obtained in the previous sections, with the risk of making an anomaly appear significant when instead its premium is null or, alternatively, making it statistically irrelevant when instead its effect is non-zero. Therefore, the objective of this section is to extend our methodology and robustify our inferential results to the case of a generic deviation from exact pricing of unknown form, i.e., global misspecification.

Consider the asset pricing restriction in (6) and assume now that, beyond the presence of

[26] A different form of misspecification, not explored in this paper, occurs when one (or more) vector of betas is a linear combination of the other ones, implying that $\mathbf{X}$ is not full-column rank. This happens, for example, when one or more of the candidate risk factors has zero (or almost zero) betas, a situation which is often referred to as the issue of *spurious* or *useless* factors. See, e.g., Jagannathan and Wang (1998), Kan and Zhang (1999b,a), Kleibergen (2009), Gospodinov, Kan, and Robotti (2014), Bryzgalova (2014), Burnside (2016), Ahn, Horenstein, and Wang (2018), Kleibergen and Zhan (2014, 2020), and Anatolyev and Mikusheva (2020), among others. The less restrictive cases of semi-strong, when $\mathbf{B}'\mathbf{B}/N = o(1)$ (see Connor and Korajczyk (2022)), and weak factors, when $\mathbf{B}'\mathbf{B} = O(1)$ (see Lettau and Pelger (2020) and Giglio, Xiu, and Zhang (2021)), are also ruled out by our assumptions. Kim, Raponi, and Zaffaroni (2020) develop an inferential procedure to test for spurious and weak factors, valid when N is large and T is fixed.

[27] Global misspecification has been studied widely in the large-T sampling scheme; see Jagannathan and Wang (1998), Shanken and Zhou (2007), Hou and Kimmel (2006), and Kan, Robotti, and Shanken (2013), among others. Gagliardini, Ossola, and Scaillet (2016) and Raponi, Robotti, and Zaffaroni (2020) show how to robustify their risk premia estimator to global misspecification in the large-N–large-T and in the large-N–fixed-T settings, respectively.

anomalies, there is a further deviation from exact pricing, due to potential global misspecification. That is, assume that:

$$\mathrm{E}[R_{i,t}|I_{t-1},\mathbf{\Pi}] = \tilde{\gamma}_{0,t-1} + \tilde{\boldsymbol{\delta}}'_{\mathrm{f},t-1}\boldsymbol{\beta}_i + \tilde{\boldsymbol{\gamma}}'_{z,t-1}\mathbf{z}_{i,t-1} + \mathrm{m}_{i,t-1}, \tag{48}$$

where $\mathrm{m}_{i,t-1}$ represents an additional pricing error, accounting for the fact the postulated model could potentially specify the wrong set of variables. In other words, one could think that the overall deviation from exact pricing in (3) has now a semiparametric structure, with the parametric part being linear in the $\mathbf{z}_{i,t-1}$, and a non-parametric component coming from the misspecification error $\mathrm{m}_{i,t-1}$, which is completely unspecified. Our objective is to test for the statistical relevance of the anomalies $\mathbf{z}_{i,t-1}$, *regardless* of whether they represent or not the full set of variables describing the true asset pricing model, that is regardless of whether $\mathrm{m}_{i,t-1}$ is zero or not.

Although we do not impose any parameterization on $\mathrm{m}_{i,t-1}$, simple considerations suggest that $\mathrm{m}_{i,t-1}$ might be cross-sectionally correlated with $\epsilon_{i,s}$, for every $s \leq t$. As an illustrative example, consider the case where one omits some relevant risk factors and anomaly variables from the true model (7), and no other sources of misspecification are present. In this circumstance, the asset-pricing model can be written as

$$\begin{aligned}\mathbf{R}_t &= \mathbf{Z}_{t-1}\tilde{\boldsymbol{\gamma}}_{\mathrm{z},t-1} + \mathbf{X}\tilde{\mathbf{\Gamma}}_{t-1} + \boldsymbol{\epsilon}_t, \quad \text{with} \\ \boldsymbol{\epsilon}_t &= \breve{\boldsymbol{\epsilon}}_t + \breve{\mathbf{Z}}_{t-1}\breve{\boldsymbol{\gamma}}_{\mathrm{z},t-1} + \breve{\mathbf{B}}\breve{\boldsymbol{\delta}}_{\mathrm{f},t-1},\end{aligned} \tag{49}$$

where $\tilde{\mathbf{\Gamma}}_{\mathrm{f},t-1} = (\tilde{\gamma}_{0,t-1}, \tilde{\boldsymbol{\delta}}'_{\mathrm{f},t-1})'$, $\breve{\mathbf{Z}}_{t-1}$ represents the $N \times \breve{K}_{\mathrm{z}}$ set of *omitted* anomalies with corresponding premia $\breve{\boldsymbol{\gamma}}_{\mathrm{z},t-1}$, and where $\breve{\mathbf{B}}$ is the $N \times \breve{K}_{\mathrm{f}}$ matrix of loadings associated with the $\breve{K}_{\mathrm{f}}$ *omitted* risk factors $\breve{\mathbf{f}}_t$, having ex-post risk premia $\breve{\boldsymbol{\delta}}_{\mathrm{f},t-1}$. [28] Finally, $\breve{\boldsymbol{\epsilon}}_t$ represents the *genuine* asset-specific component of asset returns, coinciding with $\boldsymbol{\epsilon}_t$ in the case of correct model specification. Then, combining (48) with (49), we get

$$\mathbf{m}_{t-1} = (\mathrm{m}_{1,t-1}, \cdots, \mathrm{m}_{N,t-1})' = \breve{\mathbf{Z}}_{t-1}\breve{\boldsymbol{\gamma}}_{\mathrm{z},t-1} + \breve{\mathbf{B}}\breve{\boldsymbol{\delta}}_{\mathrm{f},t-1},$$

implying that $\mathbf{m}_t$ and $\boldsymbol{\epsilon}_s$ are *cross-sectionally* correlated, through either $\breve{\mathbf{Z}}_t$ or $\breve{\mathbf{B}}$, whenever $s \leq t$, unless 'of course the premia $\breve{\boldsymbol{\gamma}}_{\mathrm{z},t-1}$ and $\breve{\boldsymbol{\delta}}_{\mathrm{f},t-1}$ are null, that is when model (7) is correctly specified.

[28]For convenience, assume that $(\mathbf{D}, \mathbf{Z}_i)'(\breve{\mathbf{F}}, \breve{\mathbf{Z}}_i) = \mathbf{0}_{K+1\times\breve{K}}$, with $\breve{K} = \breve{K}_{\mathrm{f}} + \breve{K}_{\mathrm{z}}$ and that $(\mathbf{X}, \mathbf{Z}_{t-1})'(\breve{\mathbf{B}}, \breve{\mathbf{Z}}_{t-1}) = \mathbf{0}_{K+1\times\breve{K}}$. This is with only a small loss of generality because, as discussed above, the estimated time-series regression of $\mathbf{R}_t$ on $\mathbf{f}_t$ and $\mathbf{Z}_{t-1}$ can be always re-arranged so that $(\mathbf{D}, \mathbf{Z}_i)$ and $(\breve{\mathbf{F}}, \breve{\mathbf{Z}}_i)$ are made orthogonal to each other for every i. The same applies for the estimated cross-sectional regression of $\mathbf{R}_i$ on $\boldsymbol{\beta}_i$ and $\mathbf{Z}_i$, leading to orthogonality between $\mathbf{X}, \mathbf{Z}_{t-1}$ and $\breve{\mathbf{B}}, \breve{\mathbf{Z}}_{t-1}$.

The relationship in (48) implies that the parameters $\tilde{\mathbf{\Gamma}}_{\mathrm{f},t-1}$, and $\tilde{\gamma}_{\mathrm{z},t-1}$, represent the so-called *pseudo*-true values of the premia coefficients. Formally, let $\mathbf{c}_{\mathrm{z}}$ and $\mathbf{c}_{\mathrm{f}}$ denote two arbitrary vectors of dimension K_{z} and $K_{\mathrm{f}}+1$, respectively. Then, by generalizing Shanken and Zhou (2007) and Raponi, Robotti, and Zaffaroni (2020), we define the pseudo-true premia parameters

$$(\tilde{\mathbf{\Gamma}}'_{\mathrm{f},t-1}, \tilde{\gamma}'_{\mathrm{z},t-1})' = \underset{\mathbf{c}_{\mathrm{z}},\mathbf{c}_{\mathrm{f}}}{\operatorname{argmin}} \frac{1}{N}\Big(\mathrm{E}[\mathbf{R}_t|\mathbf{I}_{t-1},\mathbf{\Pi}] - \mathbf{Z}_{t-1}\mathbf{c}_{\mathrm{z}} - \mathbf{X}\mathbf{c}_{\mathrm{f}}\Big)'\Big(\mathrm{E}[\mathbf{R}_t|\mathbf{I}_{t-1},\mathbf{\Pi}] - \mathbf{Z}_{t-1}\mathbf{c}_{\mathrm{z}} + \mathbf{X}\mathbf{c}_{\mathrm{f}}\Big), \quad (50)$$

When the model is correctly specified, then $\tilde{\mathbf{\Gamma}}_{\mathrm{f},t-1} = \mathbf{\Gamma}_{\mathrm{f},t-1}$ and $\tilde{\gamma}_{\mathrm{z},t-1} = \boldsymbol{\gamma}_{t-1,z}$, that is we recover the vector of risk and anomalies' premia of Section 5.

The cross-sectional correlation between $\mathbf{m}_t$ and $\boldsymbol{\epsilon}_s$, arising as a result of global misspecification, induces further biases to the CSR OLS estimator, which nevertheless can be consistently estimated, leading to our novel misspecification-robust premia estimators $\hat{\mathbf{\Gamma}}^{*(\mathrm{m})}_{\mathrm{f},t-1} \equiv (\hat{\gamma}^{*(\mathrm{m})}_{0,t-1}, \hat{\boldsymbol{\delta}}^{*(\mathrm{m})\prime}_{\mathrm{f},t-1})'$ and $\hat{\boldsymbol{\gamma}}^{*(\mathrm{m})}_{\mathrm{z},t-1}$, defined as follows.[29]

$$\begin{bmatrix} \hat{\mathbf{\Gamma}}^{*(\mathrm{m})}_{\mathrm{f},t-1} \\ \hat{\boldsymbol{\gamma}}^{*(\mathrm{m})}_{\mathrm{z},t-1} \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{X}}'\hat{\mathbf{X}} - N(\hat{\mathbf{\Lambda}}_1 + \hat{\mathbf{\Lambda}}^{(\mathrm{m})}_{1,t-1}) & \hat{\mathbf{X}}'\mathbf{Z}_{t-1} - N\hat{\mathbf{\Lambda}}^{(\mathrm{m})}_{3,t-1} \\ \mathbf{Z}'_{t-1}\hat{\mathbf{X}} & \mathbf{Z}'_{t-1}\mathbf{Z}_{t-1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{X}}'\mathbf{R}_t - N(\hat{\mathbf{\Lambda}}_{2,t-1} + \hat{\mathbf{\Lambda}}^{(\mathrm{m})}_{2,t-1}) \\ \mathbf{Z}'_{t-1}\mathbf{R}_t \end{bmatrix}, \quad (51)$$

setting $\hat{\mathbf{\Lambda}}_1$ and $\hat{\mathbf{\Lambda}}_{2,t-1}$ are defined in (30), and where we define the additional bias-correction terms

$$\hat{\mathbf{\Lambda}}^{(\mathrm{m})}_{1,t-1} \equiv \frac{1}{N}\begin{bmatrix} \mathbf{0}'_{K_{\mathrm{f}}+1} \\ \mathbf{P}'\hat{\mathbf{\Psi}}_{\mathrm{D}\hat{\mathrm{X}}} \end{bmatrix}, \quad \hat{\mathbf{\Lambda}}^{(\mathrm{m})}_{2,t-1} \equiv \frac{1}{N}\begin{bmatrix} 0 \\ \mathbf{P}'\hat{\mathbf{\Psi}}_{\mathrm{DR}} - \hat{\sigma}^2\mathbf{P}'\hat{\mathbf{\Psi}}_{\mathrm{D}\tilde{\mathrm{D}}} \end{bmatrix}, \text{ and } \hat{\mathbf{\Lambda}}^{(\mathrm{m})}_{3,t-1} \equiv \frac{1}{N}\begin{bmatrix} \mathbf{0}'_{K_z} \\ \mathbf{P}'\hat{\mathbf{\Psi}}_{\mathrm{DZ}} \end{bmatrix}, \quad (52)$$

with $\hat{\mathbf{\Psi}}_{\mathrm{D}\hat{\mathrm{X}}} \equiv \begin{bmatrix} \mathbf{M}^{(-1)}_{\mathrm{D},t-1}\hat{\boldsymbol{\epsilon}}\hat{\mathbf{X}} \\ \mathbf{0}_{(T-t+1)\times(K_{\mathrm{f}}+1)} \end{bmatrix}$, $\hat{\mathbf{\Psi}}_{\mathrm{DZ}} \equiv \begin{bmatrix} \mathbf{M}^{(-1)}_{\mathrm{D},t-1}\hat{\boldsymbol{\epsilon}}\mathbf{Z}_{t-1} \\ \mathbf{0}_{(T-t+1)\times K_{\mathrm{z}}} \end{bmatrix}$, $\hat{\mathbf{\Psi}}_{\mathrm{DR}} \equiv \begin{bmatrix} \mathbf{M}^{(-1)}_{\mathrm{D},t-1}\hat{\boldsymbol{\epsilon}}\mathbf{R}_t \\ \mathbf{0}_{T-t+1} \end{bmatrix}$, and $\hat{\mathbf{\Psi}}_{\mathrm{D}\tilde{\mathrm{D}}} \equiv \begin{bmatrix} \mathbf{M}^{(-1)}_{\mathrm{D},t-1}\mathbb{M}_{\mathrm{D}}\boldsymbol{\imath}_{t-1,T-1} \\ \mathbf{0}_{T-t+1} \end{bmatrix}$, setting the $(t-2)\times(T-1)$ matrix $\mathbf{M}^{(-1)}_{\mathrm{D},t-1} \equiv \mathbb{M}_{11}^{-1}[\mathbf{I}_{t-2}, \mathbf{0}_{(t-2)\times(T-t+1)}]$, where $\mathbb{M}_{11}$ denotes the $(t-2)\times(t-2)$ top-left block of $\mathbb{M}_D = \mathbf{I}_{T-1} - \mathbf{D}(\mathbf{D}'\mathbf{D})^{-1}\mathbf{D}'$.[30]

The following theorem derives the asymptotic properties of our robust estimator, extending the results of Theorem 1.

[29] See Section OA.5.3 for details of the derivation of (51).

[30] We use the partition $\mathbb{M}_D = \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}$.

Theorem 5. *As $N \to \infty$, under Assumptions 1–7 and 12*

(i)

$$\hat{\mathbf{\Gamma}}^{*(\mathrm{m})}_{\mathrm{f},t-1} - \tilde{\mathbf{\Gamma}}_{\mathrm{f},t-1} = O_p\left(\frac{1}{\sqrt{N}}\right), \quad \hat{\boldsymbol{\gamma}}^{*(\mathrm{m})}_{\mathrm{z},t-1} - \tilde{\boldsymbol{\gamma}}_{\mathrm{z},t-1} = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{53}$$

(ii)

$$\sqrt{N}\begin{bmatrix} \hat{\mathbf{\Gamma}}^{*(\mathrm{m})}_{\mathrm{f},t-1} - \tilde{\mathbf{\Gamma}}_{\mathrm{f},t-1} \\ \hat{\boldsymbol{\gamma}}^{*(\mathrm{m})}_{\mathrm{z},t-1} - \tilde{\boldsymbol{\gamma}}_{\mathrm{z},t-1} \end{bmatrix} \to_d \mathcal{N}\left(\mathbf{0}_{\mathrm{K}+1}, \mathbf{L}^{-1}_{t-1}\, \mathbf{O}^{(\mathrm{m})}_{t-1}\, \mathbf{L}_{t-1}\right) \tag{54}$$

for some $\mathbf{\Omega}^{(\mathrm{m})}_{t-1}$ defined in (OA.47), *and $\mathbf{O}^{(\mathrm{m})}_{t-1} \equiv \mathbf{O}_{t-1} + \mathbf{\Omega}^{(\mathrm{m})}_{t-1}$ with $\mathbf{L}_{t-1} > 0$ and $\mathbf{O}_{t-1}$ being the same as in Theorem 1.*[31]

Proof. See Appendix OA.4.

In Theorem OA.3 of the Online Appendix we establish $\hat{\mathbf{L}}^{(\mathrm{m})}_{t-1} \to_p \mathbf{L}_{t-1}$ and $\hat{\mathbf{\Omega}}^{(\mathrm{m})}_{t-1} \to_p \mathbf{\Omega}^{(\mathrm{m})}_{t-1}$, as $N \to \infty$, under the same assumptions of Theorem 5 and $\kappa_4 = 0$, for estimators $\hat{\mathbf{L}}^{(\mathrm{m})}_{t-1}$ and $\hat{\mathbf{\Omega}}^{(\mathrm{m})}_{t-1}$.

8 Measuring Anomalies' Contribution: Cross-Sectional $\mathbf{R^2}$ Test

Despite the considerable literature on asset pricing anomalies, how much of the cross-sectional variation in expected returns is accounted for by betas and how much by anomalies is still unclear and it still represents a challenging question.

Offering a simple criterion that can answer this question and allow to conduct formal inference on (joint) anomalies' contribution is the objective of this section. Following Chordia, Goyal, and Shanken (2015), one could consider the ratios of the (cross-sectional) variance of the beta component and of the characteristics component, with respect to the overall (cross-sectional) variance of average returns, to measure their relative contribution. Specifically, suppose one has estimated the model (27) using our bias-adjusted CSR OLS estimator, hence obtaining:

$$\mathbf{R}_t = \hat{\mathbf{X}}\hat{\mathbf{\Gamma}}^{*}_{\mathrm{f},t-1} + \mathbf{Z}_{t-1}\hat{\boldsymbol{\gamma}}^{*}_{\mathrm{z},t-1} + \hat{\boldsymbol{\eta}}_t, \tag{55}$$

[31] To ease the exposition, the definition of $\mathbf{\Omega}^{(\mathrm{m})}_{t-1}$ has been relegated to the proof of the theorem (see (OA.47)).

where $\hat{\boldsymbol{\eta}}_t$ indicates the $N \times 1$ vector of residuals. Then, the fraction of the overall variance explained by the anomaly variables $\mathbf{Z}_{t-1}$ (at any point in time) would simply be

$$\hat{R}^{2(\text{bench})}_{\text{z},t-1} \equiv \frac{\hat{\boldsymbol{\gamma}}^{*\prime}_{\text{z},t-1}\mathbf{Z}'_{t-1}\mathbb{M}_{1_N}\mathbf{Z}_{t-1}\hat{\boldsymbol{\gamma}}^{*}_{\text{z},t-1}}{\mathbf{R}'_t\mathbb{M}_{1_N}\mathbf{R}_t}. \tag{56}$$

However, despite being a very simple and intuitive measure, the R-squared in (56) could lead to several problems. First, since beta and anomaly components are not necessarily orthogonal cross-sectionally, this can lead to a fraction of the cross-sectional variance explained by the betas and by the anomaly variables - expressed by the sum of the corresponding R^2 - that is jointly greater than 100%. In addition, while orthogonality between CSR residuals and the regressors (both $\hat{\mathbf{X}}$ and $\mathbf{Z}_{t-1}$) is, by construction, warranted by the conventional CSR OLS estimator in (28), this does not hold when considering our bias-adjusted estimator $\left(\hat{\boldsymbol{\Gamma}}^{*\prime}_{\text{f},t-1}, \hat{\boldsymbol{\gamma}}^{*\prime}_{\text{z},t-1}\right)'$ of (29), implying that $\hat{R}^{2(\text{bench})}_{t-1}$ is even wrongly centred.[32]

To overcome such (lack of) orthogonality issues, let us rearrange the estimated asset pricing model (55) as follows:

$$\begin{aligned}\mathbf{R}_t &= \hat{\mathbf{X}}\hat{\boldsymbol{\Gamma}}^{*}_{\text{f},t-1} + \mathbf{Z}_{t-1}\hat{\boldsymbol{\gamma}}^{*}_{\text{z},t-1} + \mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t + \mathbb{M}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t \\ &= \mathbb{P}_{\hat{\mathbf{X}}}\left(\hat{\mathbf{X}}\hat{\boldsymbol{\Gamma}}^{*}_{\text{f},t-1} + \mathbf{Z}_{t-1}\hat{\boldsymbol{\gamma}}^{*}_{\text{z},t-1} + \mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t\right) + \mathbb{M}_{\hat{\mathbf{X}}}\left(\mathbf{Z}_{t-1}\hat{\boldsymbol{\gamma}}^{*}_{\text{z},t-1} + \mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t\right) + \mathbb{M}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t.\end{aligned} \tag{57}$$

where we use the notation $\mathbb{M}_A \equiv \mathbf{I}_a - \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}' \equiv \mathbf{I}_a - \mathbb{P}_A$, with $\mathbb{P}_A \equiv \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$, for a generic matrix $\mathbf{A}$ of dimension $a \times b$ and rank $b < a$. Notice that, by construction, the three terms on the right-hand side of (57) are now mutually orthogonal, the second term reflecting the joint contribution of $\mathbf{Z}_{t-1}$. This yields our proposed R-squared test statistic:

$$\hat{R}^{2}_{\text{z},t-1} \equiv \frac{\left(\hat{\boldsymbol{\gamma}}^{*\prime}_{\text{z},t-1}\mathbf{Z}'_{t-1} + \hat{\boldsymbol{\eta}}'_t\mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\right)\mathbb{M}_{\hat{X}}\left(\mathbf{Z}_{t-1}\hat{\boldsymbol{\gamma}}^{*}_{\text{z},t-1} + \mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t\right)}{\mathbf{R}'_t\mathbb{M}_{1_N}\mathbf{R}_t}, \tag{58}$$

which satisfies $0 \leq \hat{R}^{2}_{\text{z},t-1} \leq 1$. Notice that (58) represents a meaningful quantity, which allows us to disentangle the contribution of that portion of anomalies that is *unexplained* by - i.e., orthogonal to - the loadings (through the term $\mathbb{M}_{\hat{\mathbf{X}}}\mathbf{Z}_{t-1}$), as well as the contribution that might arise from the term $\mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t$, which is now not guaranteed to be null in general.

[32] Finally, notice that when the (true) anomalies' premia $\boldsymbol{\gamma}_{\text{z},t-1}$ are zero, then $\hat{R}^{2(\text{bench})}_{\text{z},t-1}$ will converge to zero in probability. This implies that we face a boundary problem - as necessarily $\hat{R}^{2(\text{bench})}_{t-1} \geq 0$ - which could lead to a non-standard limiting distribution of the test statistic, under the null hypothesis of zero anomalies' premia.

Armed with $\hat{R}^2_{\mathrm{z},t-1}$, one could test for the null hypothesis of zero anomalies' contribution and, in case of rejection, construct an asymptotically valid confidence interval for it. This would require establishing the limiting statistical properties of $\hat{R}^2_{\mathrm{z},t-1}$, in particular its non-standard limiting distribution, occurring when $\boldsymbol{\gamma}_{\mathrm{z},t-1} = \mathbf{0}_{K_{\mathrm{z}}}$. Therefore, in the following, we derive the asymptotic distribution of $\hat{R}^2_{\mathrm{z},t-1}$ distinguishing between the two complementary cases of zero and non-zero anomalies' premia.

Theorem 6 (R^2 test of anomalies' contribution)**.** *Set the R^2 test statistic equal to*

$$\mathcal{T}^2_{\mathrm{z},t-1} \equiv N\left(\hat{R}^2_{\mathrm{z},t-1} - \frac{\hat{\boldsymbol{\eta}}_t'\mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\mathbb{M}_{\hat{\mathbf{X}}}\mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\hat{\boldsymbol{\eta}}_t + 2\,\hat{\boldsymbol{\eta}}_t'\mathbb{P}_{[\hat{\mathbf{X}},\mathbf{Z}_{t-1}]}\mathbb{M}_{\hat{\mathbf{X}}}\mathbf{Z}_{t-1}\hat{\gamma}^*_{\mathrm{z},t-1}}{\mathbf{R}_t'\mathbb{M}_{1_N}\mathbf{R}_t}\right). \tag{59}$$

Under Assumptions 1–7, as $N \to \infty$, then:

(i) *When $\boldsymbol{\gamma}_{\mathrm{z},t-1} = \mathbf{0}_{K_{\mathrm{z}}}$,*

$$\mathcal{T}_{\mathrm{z},t-1} \to_d \sum_{j=1}^{K_{\mathrm{z}}} d_{j,t-1}\chi^2_{1,j},$$

where $(\chi^2_{1,1},\cdots,\chi^2_{1,K_{\mathrm{z}}})$ are i.i.d χ^2_1-distributed random variables , and $(d_{1,t-1},\cdots,d_{K_{\mathrm{z}},t-1})$ are the K_{z} eigenvalues of the matrix

$$\left(\mathbf{L}^{-1}_{\mathrm{z},t-1}\mathbf{O}_{t-1}\mathbf{L}^{-1\prime}_{\mathrm{z},t-1}\right)^{\frac{1}{2}}\frac{\boldsymbol{\Sigma}_{\mathrm{Z\hat{X}Z},t-1}}{\sigma_{\tilde{\mathrm{R}},t}}\left(\mathbf{L}^{-1}_{\mathrm{z},t-1}\mathbf{O}_{t-1}\mathbf{L}^{-1\prime}_{\mathrm{z},t-1}\right)^{\frac{1}{2}},$$

where $\mathbf{L}_{\mathrm{z},t-1} \equiv \left[\mathbf{0}_{K_{\mathrm{z}}\times(K_{\mathrm{f}}+1)},\mathbf{I}_{K_{\mathrm{z}}}\right]\mathbf{L}_{t-1}$, with $\mathbf{L}_{t-1}$ and $\mathbf{O}_{t-1}$ defined in (OA.35)*, and where $N^{-1}\mathbf{R}_t'\mathbb{M}_{1_N}\mathbf{R}_t \to_p \sigma_{\tilde{\mathrm{R}},t} > 0$, while $N^{-1}\mathbf{Z}_{t-1}'\mathbb{M}_{\hat{\mathbf{X}}}\mathbf{Z}_{t-1} \to_p \boldsymbol{\Sigma}_{\mathrm{Z\hat{X}Z},t-1}$.*

(ii) *When $\boldsymbol{\gamma}_{t-1,z} \neq \mathbf{0}_{K_z}$,*

$$\mathcal{T}_{\mathrm{z},t-1} \to_p \infty.$$

Moreover, under the additional Assumption 13, together with $\kappa_4 = 0$, for any $0 < \alpha < 1$,

$$\Pr\left(\hat{R}^2_{\mathrm{z},t-1} - z_{\alpha/2}\left(\frac{\hat{\omega}_{\mathrm{z},t-1}}{N}\right)^{\frac{1}{2}} \leq R^2_{\mathrm{z},t-1} \leq \hat{R}^2_{\mathrm{z},t-1} + z_{\alpha/2}\left(\frac{\hat{\omega}_{\mathrm{z},t-1}}{N}\right)^{\frac{1}{2}}\right) \to (1-\alpha).$$

where $\hat{\omega}_{\mathrm{z},t-1}$ is defined in (OA.53) and represents a consistent estimator of the asymptotic covariance matrix of $\hat{R}^2_{\mathrm{z},t-1}$, $z_{\alpha/2}$ denotes the $\alpha/2$-th quantile of the standard normal distribution, and $R^2_{t-1,z}$ denotes the limit (in probability) of $\hat{R}^2_{t-1,z}$.

Proof. See Appendix OA.4.

The result of Theorem 6 resembles the limiting behavior of the Hansen and Jagannathan (2007) (HJ) distance, which is typically used to test the null hypothesis of a correctly specified stochastic discount factor (SDF), against the alternative of misspecified models. Indeed, under the null hypotheses of correct model specification and no anomalies, respectively, both the HJ and the $\hat{R}^2_{z,t-1}$ statistics show a non-standard limiting distribution, consisting of a linear combination of i.i.d chi-squares, each of them having one degree of freedom. In contrast, the conventional Normal distribution is restored for both test statistics when considering their alternative hypotheses of either model misspecification (for the HJ statistic) or priced anomalies (in the case of $\hat{R}^2_{z,t-1}$ statistics).

Practically, Theorem 6 suggests the following empirical testing procedure to assess and quantify the effect of anomalies. At first, one would test whether the contribution of the considered anomaly variables is null or not, using the limiting results of part (i), and eliminate such variables from the asset pricing model whenever they would not provide any statistically significant contribution to the cross-section of expected returns. Alternatively, if the test results to be statically significant - that is the candidate anomalies play a significant role in explaining the cross-section of expected returns - then one could construct a valid confidence interval for $\hat{R}^2_{z,t-1}$ using the results of part (ii).

Finally, we can show that analogous properties hold in the case of a cross-sectional R-squared test that uses the local average premia estimator (38), the WLS estimator in (40) and the misspecification-robust estimator defined in (51).[33]

9 Empirical Application

9.1 Data

For our empirical exercise, we use data provided by Chen and Zimmermann (2019)[34], which contains 202 predictive firm-level characteristics at the monthly frequency. The reference period is January 1986 - December 2020. For the anomalies which are not available for the entire time period, we consider the last available month. Since our theory is derived for large N, in our analysis

[33] Details are available upon request.

[34] https://www.openassetpricing.com/data/. Details on the construction of return predictors can be found in their Online Appendix https://drive.google.com/file/d/1vXRzjxYucXZV-tgLxM26fvRZ5zKvlBXH/view

we consider only predictors for which we have enough test assets (i.e. at least 20 observations) in any given time interval. This leaves us with 170 variables, which we group following the ex ante categorization of Hou et al. (2020) in six economic categories, namely *Momentum* (15 variables), *Value versus Growth* (29 variables), *Investment* (30 variables), *Profitability* (20 variables), *Intangibles* (49 variables), and *Trading Frictions* (27 variables). A detailed list of the variables is shown in Table A.1. Following Hou et al. (2020), when performing monthly cross-sectional regressions, we winsorize the regressors at the 1% - 99% levels each month to mitigate the impact of outliers. We then standardize each regressor by subtracting its cross-sectional mean and dividing by its cross-sectional standard deviation. Monthly returns are from the Center for Research in Security Prices (CRSP), while the monthly Fama-French factors are downloaded from the Kenneth French website.

9.2 Local-Average Premia Estimator Case

We begin our empirical analysis by applying our large-N methodology under the assumptions of constant premia and correct model specification. For this analysis, we use balanced panels over fixed-time windows of three years (i.e., $T = 36$). Specifically, at each point in time, we run univariate regressions over the consider time window of three years, by regressing assets returns on the market factor and each of the 170 anomalies (averaged over the specified time interval). We then shift the time window month by month over the 1986-2020 period, and obtain the rolling time series of the t-statistics associated with the anomalies' premia and R^2 of each model.

FIGURE 3 HERE

Figure 3 shows the heatmap of the t-statistics distribution obtained for each univariate model (vertical axis) and for each time window (horizontal axis). Each cell in the map represents the degree of statistical significance of the t-statistics with a different color, from gray (non-significant t-stat), to yellow (significance at only 10% level), orange (significance at 5% level), and red (significance at 1% level). The figure clearly shows significant time variation in all the anomalies, as documented by the change in colors in the t-stat of each anomaly over time. It is also possible to identify some interesting pattern among the six categories. Particularly, *Momentum* seems to be the category with the highest pricing ability (having 92% of the variables in the category with an average absolute t-stat greater than 1.96), followed by *Profitability* (67%), *Value versus Growth* (64%), *Trading*

Frictions (63%), *Investment* (45%) and *Intangibles* (42%). For these two last categories, we also find an (overall) average $|t|$-stat below 1.96, suggesting a non significant overall contribution of the two categories in explaining the cross-sectional variation of asset returns. These results aggregated at the category level are also shown in Table II, where we also report the average value of γ_z^* for each category. Overall, if we calculate the average $|t|$-stat for each anomaly across time, we find that only 57% of the anomalies are statistically significant at 5% level. This percentage reduces to 36.6% if we look at the median absolute t-value.

TABLE II HERE

Patterns across time are instead less evident in this case, even though Figure 3 identifies a slight concentration of red points along the vertical axis in the periods of 1990-1991 and in those between 2003-2005 and 2008-2009.

FIGURE 4 HERE

To measure the influence of each anomaly on the cross-section of expected returns, in Figure 4 we report the heatmap of the R_z^2 distribution across each univariate model (vertical axis) and for each time window (horizontal axis). Specifically, for each model and for each time window, we calculate R_z^2 using (58) and represent its magnitude using different colors, from gray (which denotes an R_z^2 contribution less than 1%), up to dark blue (denoting a contribution greater than 20%). As it can be easily seen from the figure, on average, most of the anomalies exhibit a very low R_z^2. Particularly, averaging the results over time, in more than 50% of the cases we find that the fraction of the total cross-sectional variation in asset returns explained by each anomaly is below 1%. Only 4% of the variables can instead explain more than 20% of the total asset variability, and almost all of these predictors belong to the *Momentum* category. *Investments* is instead the category that shows the smallest explanatory power, having 93% of the anomalies in this category with an (average) R_z^2 below 1%.

TABLE III HERE

These results are shown in Table III, where we report the average R_z^2 for each category (first column), together with the percentage of variables within each category having an R_z^2 below 1% (second column) and greater than 20% (third column). It is also worth noticing that, in this case of univariate regressions, our R-squared test coincides with the square of the t-test. Therefore, even though the economic contribution of each anomaly to the total cross-sectional variation of asset returns is very small, it is often found to be statistically significant.

TABLE IV HERE

Given these results, one could reasonably ask whether this low explanatory power is due to the presence of factor betas in the model, i.e., whether most of the total variability in the cross-section of expected returns is captured by the market betas rather than anomalies. To answer to this question, Figure 5 shows the total variance decomposition of each model as the sum of the contribution of anomalies (red), betas (green) and residual (gray) components. First, we notice that the average contribution of the market beta to the total model variability is quite constant across all categories, ranging from 4.2% in the *Value versus Growth* category, to 4.8% in the *Intangibles* and *Trading Frictions* groups. Other interesting summary statistics aggregated at the category level are also shown in Table IV, where we report the average fraction of the total variability explained by both anomalies and betas in each category (first column), together with its decomposition in the portion coming from the anomalies (second column) and from betas (third column).

FIGURE 5 HERE

Comparing these last two columns of the table, we find that for almost all the categories the contribution of anomalies seems to be smaller than the one provided by the betas. Only for the *Momentum* category, anomalies can actually capture almost two third of the total variability provided by betas and anomalies.

With the only exception of *Momentum* (in which anomalies and betas together explain 26.6% of the total model variability), however, in almost all cases most of the cross-sectional variation in asset returns seems to be explained by the residual component, suggesting that many univariate models can actually exhibit a very low predictive power, despite the good statistical significance of single anomalies.

9.3 Time-Varying Premia Estimator Case

In this section we revisit the empirical analysis of the previous section by allowing both anomalies and risk premia to vary at each point in time, using our theoretical results derived in Section 5. In that section we show that, under our large-N theory, the (constant) bias-adjusted estimator $(\hat{\mathbf{\Gamma}}^{*\prime}, \hat{\gamma}_z^{*\prime})'$ will accurately capture the local average (across T periods) of $\mathbf{\Gamma}_{t-1}$ and $\gamma_{t-1,z}$. However, given the evidence of strong time variation in the premia estimates, it would be interesting to investigate whether the results based on local averages are still confirmed when we analyze the behavior of anomalies in a "pure" time varying setting. If this is the case, then one could justify the use of *average estimators* in empirical applications, as they would provide a representative picture of the effect of anomalies in explaining the cross-sectional variation of stock returns.

We first start the analysis by considering univariate regressions of asset returns on market beta and one anomaly at the time (Section 9.3.1). We then consider the case of multivariare regressions in Section 9.3.2, where we use more than one anomaly in each regression. In both cases, at each month t, we use the market beta obtained by running a first-pass regression using a rolling window on the past two years of data (i.e., $T = 24$).

9.3.1 Univariate Analysis

To make a direct comparison with the results obtained for the case of constant premia, in this section we consider the same univariate regressions specified in the previous section, but estimated under the assumption of time variation in both premia and anomalies.

FIGURE 6 HERE

The t-statistics associated with each anomaly premium at each month in the sample is reported in the heatmap in Figure 6. Surprisingly, very different results emerge when compered with the constant case (see Figure 3). While patterns across categories almost disappear, we now find a clear structure in the distribution of the t-statistics across time. Indeed, most of the red points in the figure are concentrated in certain intervals of time. Even more interestingly, these periods of high-significance concentration seem to correspond to episodes of major financial crises (see, e.g., the early 1990-91 recession, the dot-com bubble between 1999 and 2000, the financial crises 2007-2009, and the recent stock market crash in early 2020 due to the outbreak of COVID-19 pandemic). This is also confirmed in Figure 7, where we plot the percentage of anomalies found to be significant at 5% (or lower) confidence level at each point in time. The light gray bands correspond to NBER recession dates and to various economic and financial crises. As can be easily seen from the figure, higher percentages of significant anomalies are very often found in correspondence with periods of higher uncertainty in the markets, with peak of more than 70% of significant anomalies in the 2007-2009 financial crisis. To statistically reinforce this evidence, we also run an OLS regression, where we regress the percentage of significant anomalies on a time dummy variable, equal to one if the period t corresponds to a period of crisis and zero otherwise. What we find is a quite large and positive slope coefficient (4.29) with a corresponding t-statistics of 2.75.

FIGURE 7 HERE

To summarize, this simple analysis provides evidence of a strong variability in the regression estimates, not only among different categories of anomalies, but also across different points in time. However, such time-varying signal would be lost if we focused only on *average* estimators, as we did in the previous section or as it is typically done in the literature.

All the above results have been established using simple univariate regressions. However, since univariate regressions are rarely used in empirical applications, in the next section we apply our time-varying methodology using multiple regressions, by properly choosing a (dynamic) representative set of anomalies at each point in time.

9.3.2 Multivariate Analysis

In this section, we apply our time-varying methodology using multiple cross-sectional regressions. To identify the "best" representative set of variables, at each point in time we select the six anomalies (one for each category) which have provided the highest R_z^2 in the univariate regressions of the previous section. We then use these resulting sets of variables (together with the market betas) to perform multiple cross-sectional regressions at each month. The time-varying set of anomalies is reported graphically in Figure 8, where each red point denotes the variable that has been picked in each category (vertical axis) and for each month t (horizontal axis)[35].

[35] In some cases, when extracting the balanced panel of asset returns together with the selected six anomalies, only very few observations could remain available for the analysis. To avoid this issue, in these cases we consider alternative combinations of regressors for which we have a sufficiently large number of observations ($N > 100$), and select the combination that gives us the highest (in sample) R_z^2.

FIGURE 8 HERE

To measure the predicting ability of the selected models, in Figure 9 we plot the time series of the R_z^2 statistics (Figure 9a), together with the total variance decomposition (Figure 9b) obtained in each multivariate regression. As it can be seen from Figure 9a, the portion of total cross-sectional variation of asset returns jointly explained by the anomalies clearly varies over time, ranging from a minimum of 0.7% up to 46%. Moreover, higher R_z^2 are almost always associated with periods of economic or financial crises (represented by the gray bands in the figure), confirming again the idea that anomalies matter especially in periods of highest uncertainty. The same conclusion can be also confirmed statistically. Indeed, by applying an OLS regression of the R_z^2 time series on a time dummy (equal to one when the period t coincides with a period of crisis and zero otherwise), we find again a positive slope coefficient of 1.74 with a t-stat of 2.09.

FIGURE 9 HERE

The market beta seems meaningful as well (see the green bars in Figure 9b), with an average contribution to the total variance of almost 8% and with peaks sometimes reaching 40%. Unlike anomalies, however, highest contribution of the market beta does not seem to be related to period of crises. On average, we find that anomalies and betas together can explain more than 20% of the total cross-sectional variation of asset returns. Of this (average) 20%, 60% comes from anomalies, while betas count for the remaining 40%.

Finally, for each model, we want to assess whether the joint contribution of anomalies to the overall R-squared of the model is statistically null or not. That is, our null hypothesis is $H_0 : \boldsymbol{\gamma}_z = \mathbf{0}_{K_z}$, against the alternative that at least one anomaly is different from zero, i.e., $H_1 : \boldsymbol{\gamma}_z \neq \mathbf{0}_{K_z}$, with $K_z = 6$. For this test, we use our limiting results derived in Theorem 6, part (i), where we tabulate the asymptotic distribution of the statistics $\mathcal{T}_z$ under H_0 using 10,000 random draws from six i.i.d. χ_1^2, weighted with the estimated values $(\hat{c}_1, ..., \hat{c}_6)$ obtained in each model. The time series of the p-values associated with the $\mathcal{T}_z$ statistics for each model at a given point in time is reported in Figure 10. The yellow bands in the figure represent the p-values of all the periods in which we cannot find evidence to reject the null hypothesis ($p > 0.05$). In blue, instead, we denote the p-values ≤ 0.05, i.e. all the periods in which we can reject (at 5% confidence level) the null hypothesis of a zero anomalies' contribution. Our results suggest that the (joint) contribution of the anomalies to the total R-squared of the model is statistically different from zero only in the 38% of the cases.

Even though this percentage might seem quite low, it is worth noting that, in the 71% of cases, a significant anomalies' contribution coincide again with periods of financial downturns.

10 Conclusion

We extend the two-pass methodology for estimating and testing the effect of anomalies in asset pricing models with time-varying premia. Our methodology is designed for when large cross-sections of N assets are available but the number of time-series observations T is fixed and possibly very small, but applies also when N and T are both very large. We develop the method for ordinary and weighted least-squares estimation, and consider both cases of correct specification and global misspecification of the candidate asset pricing model. Inference relies on asymptotically valid standard errors for the premia estimators, derived in closed-form. A cross-sectional R-squared test to dissect anomalies is proposed, establishing its limiting properties under the null hypothesis of no effect of anomalies and its alternative. Using a dataset of $20,000$ individual US stock returns, we find that although anomalies are statistically significant in about half the cases (out of 170 anomalies), they explain a small fraction (less than 10%) of the cross-sectional variation of expected returns. Anomalies tend to be more important during economic and financial crises.

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Appendix

A.1 Assumptions

In this section, we present the main assumptions required for the validity of our large-N asymptotic theory, without further comments (see Section OA.2 for detailed comments). All the moments below are assumed to hold conditionally on the factors $\mathbf{F}$, even if not written explicitly, and all the limits below hold as $N \to \infty$.

It is useful to recall the $N \times K_{\mathrm{z}}(T-1)$ matrix of anomalies $\mathbf{Z} \equiv (\mathbf{z}_1, ..., \mathbf{z}_N)'$, where $\mathbf{z}_i$ defines the $K_{\mathrm{z}}(T-1) \times 1$ vector $\mathbf{z}_i \equiv \left(z_{i,1}^{(1)}, \cdots, z_{i,T-1}^{(1)}, \cdots, z_{i,1}^{(K_{\mathrm{z}})}, \cdots, z_{i,T-1}^{(K_{\mathrm{z}})}\right)'$. The $N \times K_{\mathrm{z}}$ matrix of anomalies at time $t-1$ is defined as $\mathbf{Z}_{t-1} = (\mathbf{z}_{1,t-1}, \cdots, \mathbf{z}_{N,t-1})'$, while the $(T-1) \times K_{\mathrm{z}}$ matrix of anomalies specific for the i-th asset is $\mathbf{Z}_i = (\mathbf{z}_{i,1}, \cdots, \mathbf{z}_{i,T-1})'$, setting $\mathbf{z}_{i,t-1} = \left(z_{i,t-1}^{(1)}, \cdots, z_{i,t-1}^{(K_{\mathrm{z}})}\right)'$.

Assumption 2 (*risk factors and anomalies*)**.** Set $\tilde{\mathbf{Z}}_i \equiv \mathbb{M}_{1_{T-1}} \mathbf{Z}_i$, and $\mathbf{D} \equiv (\mathbf{1}_{T-1}, \mathbf{F})$. Then, for every T, the $(T-1) \times (K+1)$ matrix $\tilde{\mathbf{D}}_i = (\mathbf{D}, \tilde{\mathbf{Z}}_i)$ satisfies

$$\tilde{\mathbf{D}}_i' \tilde{\mathbf{D}}_i > 0 \quad \text{for every } i = 1, ..., N.$$

Assumption 3 (*loadings*)**.**

$$\frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\beta}_i \to \boldsymbol{\mu}_\beta \qquad \text{and} \qquad \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\beta}_i \boldsymbol{\beta}_i' \to \boldsymbol{\Sigma}_\beta,$$

such that the matrix

$$\boldsymbol{\Sigma}_{\mathrm{X}} \equiv \begin{bmatrix} 1 & \boldsymbol{\mu}_\beta' \\ \boldsymbol{\mu}_\beta & \boldsymbol{\Sigma}_\beta \end{bmatrix} > 0.$$

Assumption 4 (*asset-specific components*)**.** The $N \times 1$ vector of error terms $\boldsymbol{\epsilon}_t$ is independently and identically distributed (i.i.d.) over time with

$$\mathrm{E}[\boldsymbol{\epsilon}_t] = \mathbf{0}_N \tag{A.1}$$

and with the $N \times N$ variance-covariance matrix satisfying

$$\mathrm{Var}\,[\boldsymbol{\epsilon}_t] = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{pmatrix} \equiv \boldsymbol{\Sigma} > 0, \tag{A.2}$$

where σ_{ij} denotes the (i,j)-th element of $\boldsymbol{\Sigma}$, for every $i, j = 1, \ldots, N$, and with $\sigma_i^2 \equiv \sigma_{ii}$.

Assumption 5 (*cross-sectional moments of asset-specific components*). (i)

$$\frac{1}{N}\sum_{i=1}^{N}\left(\sigma_i^2-\sigma^2\right)=o\left(\frac{1}{\sqrt{N}}\right), \tag{A.3}$$

for some $0<\sigma^2<\infty$.

(ii)

$$\sum_{i,j=1}^{N}\mid\sigma_{ij}\mid \mathbb{1}_{\{i\neq j\}}=o\left(N\right). \tag{A.4}$$

(iii)

$$\frac{1}{N}\sum_{i=1}^{N}\mu_{4i}\to\mu_4, \tag{A.5}$$

for some $0<\mu_4<\infty$, where $\mu_{4i}\equiv E[\epsilon_{i,t}^4]$.

(iv)

$$\frac{1}{N}\sum_{i=1}^{N}\sigma_i^4\to\sigma_4, \tag{A.6}$$

for some $0<\sigma_4<\infty$.

(v)

$$\sup_i \mu_{4i}\leq C<\infty, \tag{A.7}$$

for a generic constant C.

(vi)

$$E[\epsilon_{i,t}^3]=0. \tag{A.8}$$

(vii)

$$\frac{1}{N}\sum_{i=1}^{N}\kappa_{4,iiii}\to\kappa_4, \tag{A.9}$$

for some $0\leq|\kappa_4|<\infty$, where $\kappa_{4,iiii}\equiv\kappa_4[\epsilon_{it},\epsilon_{it},\epsilon_{it},\epsilon_{it}]$ denotes the fourth-order cumulant of the asset-specific component $\{\epsilon_{i,t},\epsilon_{i,t},\epsilon_{i,t},\epsilon_{i,t}\}$.

(viii) For every $3\leq h\leq 8$, all the following mixed cumulants of order h satisfy

$$\sup_{i_1}\sum_{i_2,\ldots,i_h=1}^{N}|\kappa_{h,i_1i_2\ldots i_h}|=o\left(N\right), \tag{A.10}$$

for at least one i_j $(2\leq j\leq h)$ different from i_1, where $\kappa_{h,i_1,i_2\ldots i_h}$ is the mixed cumulant in the $\{\epsilon_{i_1,s},\epsilon_{i_2,s},\cdots,\epsilon_{i_h,s}\}$ of order h.

Assumption 6 (*CLT of asset-specific component*). (i)

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i \rightarrow_d \mathcal{N}\left(\mathbf{0}_{T-1}, \sigma^2\mathbf{I}_{T-1}\right). \tag{A.11}$$

(ii)

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\text{vec}\left(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - \sigma_i^2\mathbf{I}_{T-1}\right) \rightarrow_d \mathcal{N}\left(\mathbf{0}_{(T-1)^2}, \mathbf{U}_\epsilon\right), \tag{A.12}$$

where $\mathbf{U}_\epsilon \equiv \lim \frac{1}{N}\sum_{i,j=1}^{N} E\Big[\text{vec}(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - \sigma_i^2\mathbf{I}_{T-1})\text{vec}(\boldsymbol{\epsilon}_j\boldsymbol{\epsilon}_j' - \sigma_j^2\mathbf{I}_{T-1})'\Big]$.

(iii) For any $T \times 1$ vector $\mathbf{c}$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\mathbf{c}' \otimes \begin{pmatrix}1\\ \boldsymbol{\beta}_i\end{pmatrix}\right)\boldsymbol{\epsilon}_i \rightarrow_d \mathcal{N}\left(\mathbf{0}_{K_\text{f}+1}, (\mathbf{c}'\mathbf{c})\sigma^2\boldsymbol{\Sigma}_\text{X}\right). \tag{A.13}$$

Remark 1. *The expression for* $\mathbf{U}_\epsilon$ *in* (A.12) *can be derived in closed form. In particular, Raponi, Robotti, and Zaffaroni (2020) established that the* $T^2 \times T^2$ *matrix* $\mathbf{U}_\epsilon$ *has the following form*

$$\mathbf{U}_\epsilon = \left[\begin{array}{c:c:c:c:c} U_{11} & \cdots & U_{1t} & \cdots & U_{1T} \\ \hdashline \vdots & \ddots & \vdots & \vdots & \vdots \\ \hdashline U_{t1} & \cdots & U_{tt} & \cdots & U_{tT} \\ \hdashline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hdashline U_{T1} & \cdots & U_{Tt} & \cdots & U_{TT} \end{array}\right].$$

Each block of U_ϵ *is a* $T \times T$ *matrix. The blocks along the main diagonal, denoted by* U_{tt}, $t = 1, 2, \ldots, T$, *are themselves diagonal matrixes with* $(\kappa_4 + 2\sigma_4)$ *in the* (t,t)*-th position and* σ_4 *in the* (s,s) *position for every* $s \neq t$. *The blocks outside the main diagonal, denoted by* U_{ts}, $s, t = 1, 2, \ldots, T$ *with* $s \neq t$, *are all made of zeros except for the* (s,t)*-th position that contains* σ_4; *that is,*

$$U_{tt} = \underset{t\text{-th row}}{\rightarrow} \begin{bmatrix} \sigma_4 & \cdots & 0 & \overset{\overset{\downarrow}{t\text{-th column}}}{\cdots} & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \sigma_4 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & (\kappa_4 + 2\sigma_4) & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \sigma_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \sigma_4 \end{bmatrix}, \quad U_{ts} = \underset{s\text{-th row}}{\rightarrow} \begin{bmatrix} 0 & \cdots & 0 & \overset{\overset{\downarrow}{t\text{-th column}}}{\cdots} & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \sigma_4 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}.$$

Assumption 7 (*moments and CLT of anomalies*)**.** *Define the* $K_{\mathrm{z}}(T-1)^2 \times 1$ *vector* $\mathbf{u}_i \equiv \boldsymbol{\epsilon}_i \otimes \mathbf{z}_i$.

(i)

$$\frac{\mathbf{Z}'\mathbf{1}_N}{N} \rightarrow_p (\boldsymbol{\mu}_{\mathrm{z}} \otimes \mathbf{1}_{T-1}) \equiv \boldsymbol{\mu}_{\mathrm{z,T-1}}$$

for a finite $K_{\mathrm{z}} \times 1$ vector $\boldsymbol{\mu}_{\mathrm{z}} = \left(\mu_{\mathrm{z}}^{(1)}, \ldots, \mu_{\mathrm{z}}^{(K_{\mathrm{z}})}\right)' \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\mu}_{\mathrm{z}_i}$, setting $\boldsymbol{\mu}_{\mathrm{z}_i} \equiv \mathrm{E}[\mathbf{z}_{i,s}]$.

(ii)

$$\frac{\mathbf{Z}'\mathbf{Z}}{N} \rightarrow_p \boldsymbol{\Sigma}_{\mathrm{Z}},$$

for a finite $K_{\mathrm{z}}(T-1) \times K_{\mathrm{z}}(T-1)$ matrix $\boldsymbol{\Sigma}_{\mathrm{Z}}$, such that $\mathbb{J}'\boldsymbol{\Sigma}_{\mathrm{Z}}\mathbb{J} > 0$ and $\mathbb{J}_{t-1}'\boldsymbol{\Sigma}_{\mathrm{Z}}\mathbb{J}_{t-1} > 0$, for every $2 \leq t \leq T$.

(iii)

$$\frac{\mathbf{Z}'\mathbf{B}}{N} \rightarrow_p \boldsymbol{\Sigma}_{\mathrm{ZB}},$$

for a finite $K_{\mathrm{z}}(T-1) \times K_{\mathrm{f}}$ matrix $\boldsymbol{\Sigma}_{\mathrm{ZB}}$.

(iv) *Setting* $\boldsymbol{\mu}_{\mathrm{u}_i} \equiv \mathrm{E}[\mathbf{u}_i]$,

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\mu}_{\mathrm{u}_i} = o\left(\frac{1}{\sqrt{N}}\right).$$

(v) *Setting* $\boldsymbol{\Sigma}_{\mathrm{u},ij} \equiv \mathrm{Cov}[\mathbf{u}_i, \mathbf{u}_j]$, *for* $i, j = 1, ..., N$,

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\Sigma}_{\mathrm{u},ii} \rightarrow \boldsymbol{\Sigma}_{\mathrm{U}} \equiv (\sigma^2 \mathbf{I}_{T-1} \otimes \boldsymbol{\Sigma}_{\mathrm{Z}}) \text{ and } \sum_{i,j=1}^{N} \boldsymbol{\Sigma}_{\mathrm{u},ij} \mathbb{1}_{i\neq j} = o(N).$$

(vi) For any $i, j = 1, ..., N$,

$$\mathrm{Cov}\left[\mathbf{z}_{i,t}, \boldsymbol{\epsilon}_j' \otimes \boldsymbol{\epsilon}_j'\right] = \mathbf{0}_{K_{\mathrm{z}}\times(T-1)^2}, \quad \mathrm{Cov}\left[\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_j' \otimes (\boldsymbol{u}_j - \mathrm{E}[\boldsymbol{u}_j])'\right] = \mathbf{0}_{T-1\times K_{\mathrm{z}}(T-1)^3}.$$

(vii)

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} (\boldsymbol{u}_i - \boldsymbol{\mu}_{u_i}) \rightarrow_d \mathcal{N}\left(\mathbf{0}_{K_{\mathrm{z}}(T-1)^2}, \boldsymbol{\Sigma}_{\mathrm{U}}\right).$$

(viii) *Setting* $\mathbf{\Sigma}_{\mathrm{u}\epsilon,ij} \equiv \mathrm{Cov}\left[\boldsymbol{\epsilon}_i \otimes \boldsymbol{\epsilon}_i, \mathbf{u}_j'\right]$,

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{\Sigma}_{\mathrm{u}\epsilon,ii} \quad \rightarrow \quad \mathbf{\Sigma}_{\mathrm{u}\epsilon} = \mathbf{0}_{(T-1)^2 \times K_{\mathrm{z}}(T-1)^2} \text{ and } \frac{1}{N}\sum_{i,j=1}^{N} \mathbf{\Sigma}_{\mathrm{u}\epsilon,ij} \rightarrow \mathbf{0}_{(T-1)^2 \times K_{\mathrm{z}}(T-1)^2}.$$

(ix)

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \mathrm{Cov}\left[\boldsymbol{u}_i \otimes \boldsymbol{u}_i, \boldsymbol{u}_j' \otimes \boldsymbol{u}_j'\right] \rightarrow \mathbf{0}_{K_{\mathrm{z}}^2(T-1)^4 \times K_{\mathrm{z}}^2(T-1)^4}.$$

(x) *Let* $\mathbb{P}_{\tilde{Z}_i} = \tilde{\mathbf{Z}}_i(\tilde{\mathbf{Z}}_i'\tilde{\mathbf{Z}}_i)^{-1}\tilde{\mathbf{Z}}_i'$, *with its generic* (t,s) *element denoted by* $\mathbb{p}_{i,ts}$, *for* $t,s = 1,...,T-1$, *where* $\tilde{\mathbf{Z}}_i = \mathbb{M}_{1_{T-1}}\mathbf{Z}_i$. *Then, for every* $1 \leq t+1, s+1, v_a, u_a \leq (T-1)$, *with* $a = 1,...,4$, *the following hold:*

$$(x.1) \qquad \frac{1}{N}\sum_{i=1}^{N} \mathbb{P}_{\tilde{Z}_i} \rightarrow_p \mathbb{P}_{\tilde{Z}}, \text{ for a finite matrix } \mathbb{P}_{\tilde{Z}},$$

$$(x.2) \qquad \frac{1}{N}\sum_{i=1}^{N} (\mathbb{P}_{\tilde{Z}_i} \odot \mathbb{P}_{\tilde{Z}_i}) \rightarrow_p \mathbb{P}_{\tilde{Z}}^{(2)}, \text{ for a finite matrix } \mathbb{P}_{\tilde{Z}}^{(2)},$$

$$(x.3) \qquad \frac{1}{N}\sum_{i=1}^{N} \mathbb{P}_{\tilde{Z}_i}(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - \sigma_i^2\mathbf{I}_{T-1}) = \mathbb{P}_{\tilde{Z}}\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i' - \sigma_i^2\mathbf{I}_{T-1}) + o_p\left(\frac{1}{\sqrt{N}}\right),$$

$$(x.4) \qquad \frac{1}{N^2}\sum_{i,j=1}^{N} \kappa_4\left[\prod_{a=1}^{4}\mathbb{p}_{i,t-1u_a}, \prod_{a=1}^{4}\mathbb{p}_{j,s-1v_a}, \prod_{a=1}^{4}\epsilon_{i,u_a+1}, \prod_{a=1}^{4}\epsilon_{j,v_a+1}\right] = o(1),$$

$$(x.5) \qquad \frac{1}{N^2}\sum_{i,j=1}^{N} \kappa_3\left[\prod_{a=1}^{4}\mathbb{p}_{i,t-1u_a}, \prod_{a=1}^{4}\mathbb{p}_{j,s-1v_a}, \prod_{a=1}^{4}\epsilon_{i,u_a+1}\right] = o(1),$$

$$(x.6) \qquad \frac{1}{N^2}\sum_{i,j=1}^{N} \kappa_3\left[\prod_{a=1}^{4}\mathbb{p}_{i,t-1u_a}, \prod_{a=1}^{4}\epsilon_{i,u_a+1}, \prod_{a=1}^{4}\epsilon_{j,v_a+1}\right] = o(1),$$

$$(x.7) \qquad \frac{1}{N^2}\sum_{i,j=1}^{N} \mathrm{Cov}\left[\prod_{a=1}^{4}\mathbb{p}_{i,t-1u_a}, \prod_{a=1}^{4}\epsilon_{j,v_a+1}\right] = o(1),$$

$$(x.8) \qquad \frac{1}{N^2}\sum_{i,j=1}^{N} \mathrm{Cov}\left[\mathbb{p}_{j,su_1}\mathbb{p}_{i,tv_1}, \epsilon_{i,t+1}\epsilon_{j,s+1}\epsilon_{iu_1+1}\epsilon_{jv_1+1}\right] = o(1),$$

$$(x.9) \qquad \frac{1}{N}\sum_{i=1}^{N} \mathrm{Cov}\left[\prod_{a=1}^{4}\mathbb{p}_{i,t-1u_a}, \prod_{a=1}^{4}\epsilon_{i,v_a+1}\right] = o(1).$$

where $\kappa_3[\cdot,\cdot,\cdot]$ and $\kappa_4[\cdot,\cdot,\cdot,\cdot]$ denote the mixed cumulants of order 3 and 4, respectively.

(xi) For every $3 \leq h \leq 8$, all the following mixed cumulants of order h satisfy

$$\sup_{i_1} \sum_{i_2,\ldots,i_h=1}^{N} |\kappa^{\mathbb{P}}_{h,i_1 i_2 \ldots i_h}| = o\left(N\right), \tag{A.14}$$

for at least one i_j $(2 \leq j \leq h)$ different from i_1, where $\kappa^{\mathbb{P}}_{h,i_1,i_2 \ldots i_h}$ is the mixed cumulant in the $\{\mathbb{p}_{i_1,t_1-1 u_1}, \mathbb{p}_{i_2,t-2 1 u_2}, \cdots, \mathbb{p}_{i_h,t_h-1 u_h},\}$ of order h, for every $2 \leq t_1, \cdots, t_h, u_1, \cdots, u_h \leq T$.

A.1.1 Additional assumptions required for the WLS estimation

In this Section, we introduce additional assumptions that are required for the validity of the WLS estimation described in Section 6. Before stating the main assumptions, it is useful to introduce some preliminary notation. In the following, we denote by $\mathbf{w}_{i.} \equiv (\mathrm{w}_{i,1}, \cdots, \mathrm{w}_{i,T-1})'$ the $(T-1) \times 1$ vector of weights specific for the i-th asset, and by $\mathbf{w}_{.t-1} \equiv (\mathrm{w}_{1,t-1}, \cdots, \mathrm{w}_{N,t-1})'$ the $N \times 1$ vector of weights at time $(t-1)$, for every $2 \leq t \leq T$, with the $N \times T$ matrix $\mathbf{W} = (\mathbf{w}_{.1}, \cdots, \mathbf{w}_{.T-1}) = (\mathbf{w}_{1.}, \cdots, \mathbf{w}'_{N.})$.

Assumption 8. *(CSR WLS weights)*

(i)

$$\frac{\mathbf{1}'_N \mathbf{W}_{t-1} \mathbf{1}_N}{N} \to_p 1.$$

(ii) For any real number $h > 1$ then,

$$\frac{\mathbf{1}'_N \mathbf{W}^h_{t-1} \mathbf{1}_N}{N} \to_p \mu^h_{\mathrm{w},t-1}$$

(iii)

$$\frac{1}{N} \sum_{i=1}^{N} \mathbf{w}_{i.} \mathbf{w}'_{i.} \to_p \mathbf{\Sigma}_{\mathrm{W}}.$$

Assumption 9. *(Weighted loadings)* Let $\mathbf{W}_{t-1}$ satisfy Assumption 8 and let the loadings $\boldsymbol{\beta}_i$ be a non-random sequence. As $N \to \infty$, then

$$\frac{1}{N}\mathbf{B}'\mathbf{W}_{t-1}\mathbf{1}_N \to_p \boldsymbol{\mu}_\beta \qquad \text{and} \qquad \frac{1}{N}\mathbf{B}'\mathbf{W}_{t-1}\mathbf{B} \to_p \mathbf{\Sigma}_\beta, \tag{A.15}$$

such that

$$\mathbf{\Sigma}_{\mathrm{X}} \equiv \begin{bmatrix} 1 & \boldsymbol{\mu}'_\beta \\ \boldsymbol{\mu}_\beta & \mathbf{\Sigma}_\beta \end{bmatrix} > 0. \tag{A.16}$$

Assumption 10. *(Weighted cross-sectional moments of returns' innovations)* As $N \to \infty$,

(i) Let $0 < \sigma^2 < \infty$.Then, for every $2 \leq t \leq T$:

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{w}_{i,t-1}\left(\sigma_i^2 - \sigma^2\right) = o_p\left(\frac{1}{\sqrt{N}}\right), \tag{A.17}$$

(ii)

$$\sum_{i,j=1}^{N} \mathrm{w}_{i,t-1} \mid \sigma_{ij} \mid \mathbb{1}_{\{i \neq j\}} = o_p\left(N\right). \tag{A.18}$$

(iii) Let $0 < \mu_4 < \infty$, and let $\mu_{4i} = \mathrm{E}[\epsilon_{it}^4]$. Then, for every $2 \leq t \leq T$:

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{w}_{i,t-1}\mu_{4i} \to_p \mu_4, \tag{A.19}$$

(iv) Let $0 < \sigma_4 < \infty$. Then, for every $2 \leq t \leq T$:

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{w}_{i,t-1}\sigma_i^4 \to_p \sigma_4, \tag{A.20}$$

(v) *Let $\kappa_3(a, b, c)$ denote the third-order cumulant of the random variables $a, b,$ and c. Then,*

$$\kappa_3[\epsilon_{i,t}, \epsilon_{j,s}, \mathrm{w}_{j,h}] = 0, \quad \textit{and} \quad \kappa_3[\epsilon_{i,t}, \epsilon_{j,s}, \mathbf{z}_{j,h}] = \mathbf{0}_{K_z}. \tag{A.21}$$

(vi) Let $\kappa_{4,iiii} = \kappa_4[\epsilon_{i,t}, \epsilon_{i,t}, \epsilon_{i,t}, \epsilon_{i,t}]$ denote the fourth-order cumulant of the asset-specific error $\{\epsilon_{i,t}, \epsilon_{i,t}, \epsilon_{i,t}, \epsilon_{i,t}\}$. Then, for some $0 \leq |\kappa_4| < \infty$ and for every $2 \leq t \leq T$:

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{w}_{i,t-1}\kappa_{4,iiii} \to_p \kappa_4. \tag{A.22}$$

(vii) For every $3 \leq h \leq 8$, all the following mixed cumulants of order h satisfy

$$\sup_{i_1} \sum_{i_2,\ldots,i_h=1}^{N} |\kappa_{h,\mathrm{w}_{i_1,t-1} i_2 \ldots i_h}| = o\left(N\right), \tag{A.23}$$

and

$$\sup_{i_1} \sum_{i_2,\ldots,i_h=1}^{N} |\kappa_{h\mathrm{w}_{i_1,t-1},\mathbf{z}_{i_2,r}, i_3 \ldots i_h}| = o\left(N\right), \tag{A.24}$$

for at least one i_j $(2 \leq j \leq h)$ different from i_1, where $\kappa_{h,\mathrm{w}_{i_1,t-1} i_2 \ldots i_h}$ is the mixed cumulant in the $\{\mathrm{w}_{i_1,t-1}, \epsilon_{i_2,s}, \cdots, \epsilon_{i_h,s}\}$ of order h, and $\kappa_{h,\mathrm{w}_{i_1,t-1},\mathbf{z}_{i_2,r}, i_3 \ldots i_h}$ is the mixed cumulant in the $\{\mathrm{w}_{i_1,t-1}, \mathbf{z}_{i_2,r}, \epsilon_{i_3,s}, \cdots, \epsilon_{i_h,s}\}$ of order h.

Assumption 11. *(Weighted moments and CLT of anomalies) We define the* $(T-1)^2 \times 1$ *vector* $\mathbf{v}_i \equiv (\boldsymbol{\epsilon}_i \otimes \mathbf{w}_i)$ *and the corresponding* $N \times (T-1)^2$ *matrix* $\mathbf{V} \equiv (\mathbf{v}_1, \cdots, \mathbf{v}_N)'$*, such that* $\mathrm{E}[\mathbf{v}_i] \equiv \boldsymbol{\mu}_{\mathrm{v}_i} < \infty$*, and* $\boldsymbol{\Sigma}_{\mathrm{v},ij} \equiv \mathrm{Cov}\left[\mathbf{v}_i, \mathbf{v}_j\right]$*.*

(i)

$$\frac{\boldsymbol{\epsilon}\left(\mathbf{W}_{t-1} - \mathrm{E}[\mathbf{W}_{t-1}]\right)\boldsymbol{\epsilon}'}{N} \to_p \mathbf{0}_{(T-1)\times(T-1)}.$$

(ii)

$$\frac{\mathbf{Z}_{t-1}'\mathbf{W}_{t-1}\mathbf{1}_N}{N} \quad \to_p \quad \boldsymbol{\mu}_{\mathrm{z},t-1} \; \textit{and} \; \frac{\mathbf{Z}_{t-1}'\mathbf{W}_{t-1}\mathbf{Z}_{t-1}}{N} \to_p \boldsymbol{\Sigma}_{\mathrm{Z},t-1}.$$

(iii) *Let* $\boldsymbol{\Sigma}_{\mathrm{ZW}}$ *be a finite* $K_{\mathrm{z}}(T-1) \times (T-1)$ *matrix. Then,*

$$\frac{\mathbf{Z}'\mathbf{W}}{N} \to_p \boldsymbol{\Sigma}_{\mathrm{ZW}}.$$

(iv)

$$\frac{1}{N}\left(\mathbf{Z}_{t-1} - \mathrm{E}[\mathbf{Z}_{t-1}]\right)'\left(\mathbf{W}_{t-1} - \mathrm{E}[\mathbf{W}_{t-1}]\right)\boldsymbol{\epsilon}' \to_p \mathbf{0}_{K_{\mathrm{z}}\times(T-1)}.$$

(v)

$$\frac{1}{N}\left(\mathbf{Z}_{t-1} - \mathrm{E}[\mathbf{Z}_{t-1}]\right)'\mathbf{W}_{t-1}\boldsymbol{\epsilon}' - \frac{1}{N}\left(\mathbf{Z}_{t-1} - \mathrm{E}[\mathbf{Z}_{t-1}]\right)'\boldsymbol{\epsilon}' = o_p\left(N^{-\frac{1}{2}}\right).$$

(vi)

$$\frac{\mathbf{X}'\mathbb{M}_{1_N}\mathbf{V}}{N} = o_p\left(N^{-\frac{1}{2}}\right), \quad \text{and} \quad \frac{\mathbf{Z}'\mathbb{M}_{1_N}\mathbf{V}}{N} = o_p\left(N^{-\frac{1}{2}}\right).$$

(vii)

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathrm{v},ii} \to \boldsymbol{\Sigma}_{\mathrm{V}} \equiv \sigma^2\mathbf{I}_{T-1}\otimes\boldsymbol{\Sigma}_{\mathrm{W}}, \quad \text{and} \quad \sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathrm{v},ij}\mathbb{1}_{i\neq j} = o(N)$$

(viii)

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\mathbf{v}_i - \boldsymbol{\mu}_{\mathrm{v}_i}\right) \to_d N\left(\mathbf{0}_{(T-1)^2}, \boldsymbol{\Sigma}_{\mathrm{V}}\right) \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\mu}_{\mathrm{v_i}} = o\left(N^{-\frac{1}{2}}\right).$$

(ix)

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\mathbf{z}_i - \boldsymbol{\mu}_{\mathrm{z}_i}) \to_d N(\mathbf{0}_{K_z(T-1)}, \boldsymbol{\Sigma}_{ZZ}) \; \textit{and} \; \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\mu}_{\mathrm{z}_i} - \boldsymbol{\mu}_z) = o(N^{-\frac{1}{2}}).$$

A.1.2 Additional assumptions required for estimation under model misspecification

Assumption 12. *(mixed-moments of pricing errors)*

(i)

$$\frac{1}{N}\boldsymbol{\epsilon}\mathbf{m}_{t-1} \to_p \begin{bmatrix}\boldsymbol{\theta}_{t-1,m}\\ \mathbf{0}_{T-t+1}\end{bmatrix},$$

with $\boldsymbol{\theta}_{t-1,m} \equiv (\theta_{t-3,m}, \theta_{t-4,m}, \ldots, \theta_{0,m})'$, defined as, for every $2 \leq s, t \leq T$,

$$\frac{1}{N}\sum_{i=1}^{N}\epsilon_{i,s}m_{i,t-1} \to_p \theta_{t-1-s,m}, \text{ such that } \theta_{u,m} = 0 \text{ for } u < 0.$$

(ii)

$$\frac{1}{N}\mathbf{m}'_{t-1}\mathbf{m}_{t-1} \to_p \sigma_{t-1mm}.$$

(iii)

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{P}_{\tilde{D}_i}\boldsymbol{\epsilon}_i m_{i,t-1} \to_p \mathbf{0}_{T-1}.$$

A.1.3 Additional assumptions required for the cross-sectional R-squared test

In this Section we introduce additional assumptions that are required to derive the R-squared test described in Section 8.

Assumption 13. (i)

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\beta}_i - \boldsymbol{\mu}_\beta = o\left(N^{-\frac{1}{2}}\right) \text{ and } \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\beta}_i\boldsymbol{\beta}_i' - \boldsymbol{\Sigma}_\beta = o\left(N^{-\frac{1}{2}}\right).$$

(ii)

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left((\mathbf{z}_i\otimes\mathbf{z}_i) - \text{vec}(\boldsymbol{\Sigma}_{\text{Z}})\right) \to_d N(\mathbf{0}_{K_z^2}, \mathbf{U}_{\text{Z}}), \quad \text{with}$$

$$\frac{1}{N}\sum_{i=1}^{N}\text{E}\left[(\mathbf{z}_i\otimes\mathbf{z}_i) - \text{vec}(\boldsymbol{\Sigma}_{\text{Z}})\right] = o\left(N^{-\frac{1}{2}}\right), \; \frac{1}{N}\sum_{i=1}^{N}\text{E}\left[(\mathbf{z}_i\otimes\mathbf{z}_i) - \text{vec}(\boldsymbol{\Sigma}_{\text{Z}})\right]\left[(\mathbf{z}_i\otimes\mathbf{z}_i) - \text{vec}(\boldsymbol{\Sigma}_{\text{Z}})\right]' \to \mathbf{U}_{\text{Z}},$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N}\text{E}\left[(\mathbf{z}_i\otimes\mathbf{z}_i) - \text{vec}(\boldsymbol{\Sigma}_{\text{Z}})\right]\left[(\mathbf{z}_j\otimes\mathbf{z}_j) - \text{vec}(\boldsymbol{\Sigma}_{\text{Z}})\right]' = o(N), \text{ and } \frac{1}{N}\sum_{i,j=1}^{N}\text{Cov}\left[(\mathbf{z}_i\otimes\mathbf{z}_i), \mathbf{z}_j'\right] \to \boldsymbol{\Sigma}_{\text{z}\otimes\text{z}}.$$

(iii)

$$\sqrt{N}\left(\frac{\mathbf{Z}'\mathbf{1}_N}{N}-\boldsymbol{\mu}_{\mathrm{z},T-1}\right)\rightarrow_d N\left(\mathbf{0}_{K_{\mathrm{z}}(T-1)},\boldsymbol{\Sigma}_{\mathrm{Z}}-\boldsymbol{\mu}_{\mathrm{z},T-1}\boldsymbol{\mu}'_{\mathrm{z},T-1}\right).$$

(iv)

$$\frac{1}{N}\sum_{i,j=1}^{N}\mathrm{Cov}\left((\mathbf{z}_i\otimes\mathbf{z}_i),(\boldsymbol{\epsilon}'_j\otimes\boldsymbol{\epsilon}'_j)\right)\rightarrow\boldsymbol{\Sigma}_{\mathrm{Z}\otimes\epsilon}=\mathbf{0}_{((T-1)K_z)^2\times(T-1)^2}.$$

(v)

$$\frac{1}{N}\sum_{i,j=1}^{N}\mathrm{Cov}\left((\mathbf{z}_i\otimes\mathbf{z}_i),\mathbf{u}'_j\right)\rightarrow\boldsymbol{\Sigma}_{\mathrm{ZU}}=\mathbf{0}_{((T-1)K_z)^2\times(T-1)^2K_z}.$$

A.2 Empirics: Tables and Plots

Table II: **Estimation results by categories - constant case**. The table shows the average estimate of γ_z^* (first column) and the average $|t|$-statistic (second column) across each category, together with the percentage of anomalies within each category found to be statistically significant at 5% confidence level (third column). The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section OA.5.2. The results are then averaged across time and aggregated at the category level. The analysis uses balanced panels over fixed-time windows of three years, with a reference period ranging from January 1986 to December 2020.

Category	Average γ_z^* (median)	Average $|t|$-stat (median)	$|t| > 1.96$ (%)
Momentum	2.36 (1.95)	11.46 (11.69)	92%
Value versus Growth	-0.22 (-0.08)	3.56 (2.84)	64%
Investment	0.14 (0.10)	1.29 (1.10)	45%
Profitability	0.11 (0.10)	2.18 (2.16)	67%
Intangibles	0.36 (0.043)	1.82 (1.58)	42%
Trading Frictions	-0.14 (0.003)	2.24 (2.09)	63%

Table III: **Anomalies contribution by categories - constant case.** The table shows the average R_z^2 contribution of each category (first column), together with the percentage of anomalies in each category having a contribution below 1% (second column) and greater that 20% (third column). The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section OA.5.2. In each model, the R_z^2 statistics has been calculated using the quantity in (58). The results are then averaged across time and aggregated at the category level. The analysis uses balanced panels over fixed-time windows of three years, with a reference period ranging from January 1986 to December 2020.

Category	Average R_z^2 (%)	$R_z^2 < 1\%$	$R_z^2 > 20\%$
Momentum	21.9	8%	33%
Value versus Growth	2.8	32%	0%
Investment	0.7	93%	0%
Profitability	1.2	61%	0%
Intangibles	3.5	61%	2%
Trading Frictions	1.6	31%	0%

Table IV: **R-squared decomposition by categories - constant case.** The table shows the average fraction of the total model variability explained by both anomalies and market betas in each category (first column), together with its decomposition in the (average) portion coming from anomalies (second column) and from betas (third column). The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section OA.5.2 of constant premia. The results are then averaged across time and aggregated at the category level. The analysis uses balanced panels over fixed-time windows of three years, with a reference period ranging from January 1986 to December 2020.

Category	% explained betas + anomalies	of which from anomalies	of which from betas
Momentum	26.6	63.8%	36.2%
Value versus Growth	6.9	31.6%	68.4%
Investment	5.1	11.4%	88.6%
Profitability	5.8	18.4%	81.6%
Intangibles	8.3	20.1%	79.9%
Trading Frictions	6.4	23.1%	76.9%

Figure 3: **Heatmap of the t-statistics distribution - constant case**. The figure shows the heatmap of the t-statistics distribution obtained in each of the 170 univariate model (vertical axis) and for each time window (horizontal axis), with $T = 36$ months. Each cell in the map represents the degree of statistical significance of the t-statistics with a different color, from gray (non-significant t-stat), to yellow (significance at only 10% level), orange (significance at 5% level), and red (significance at 1% level). The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section OA.5.2. The analysis uses balanced panels over fixed-time windows of three years, with a reference period ranging from January 1986 to December 2020.

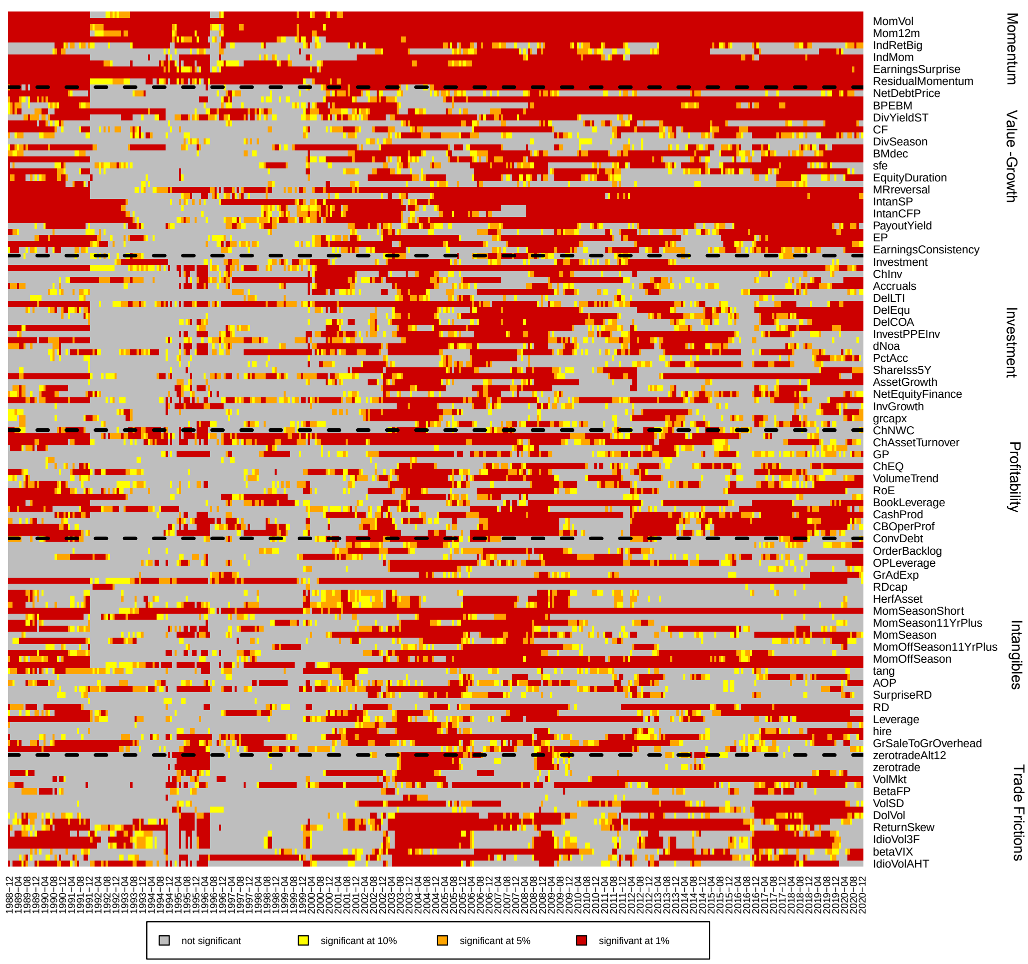

Figure 4: **Heatmap of the R_z^2 distribution - constant case**. The figure shows the heatmap of the R_z^2 distribution obtained in each of the 170 univariate model (vertical axis) and for each time window (horizontal axis), with $T = 36$ months. Each cell in the map represents the fraction of the total cross-sectional variation in asset returns explained by each anomaly, using different colors, from gray (R^2 contribution less than 1%) up to dark blue (R^2 contribution greater than 20%. The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section OA.5.2. In each model, the R_z^2 statistics has been calculated using the quantity in (8). The analysis uses balanced panels over fixed-time windows of three years, with a reference period ranging from January 1986 to December 2020.

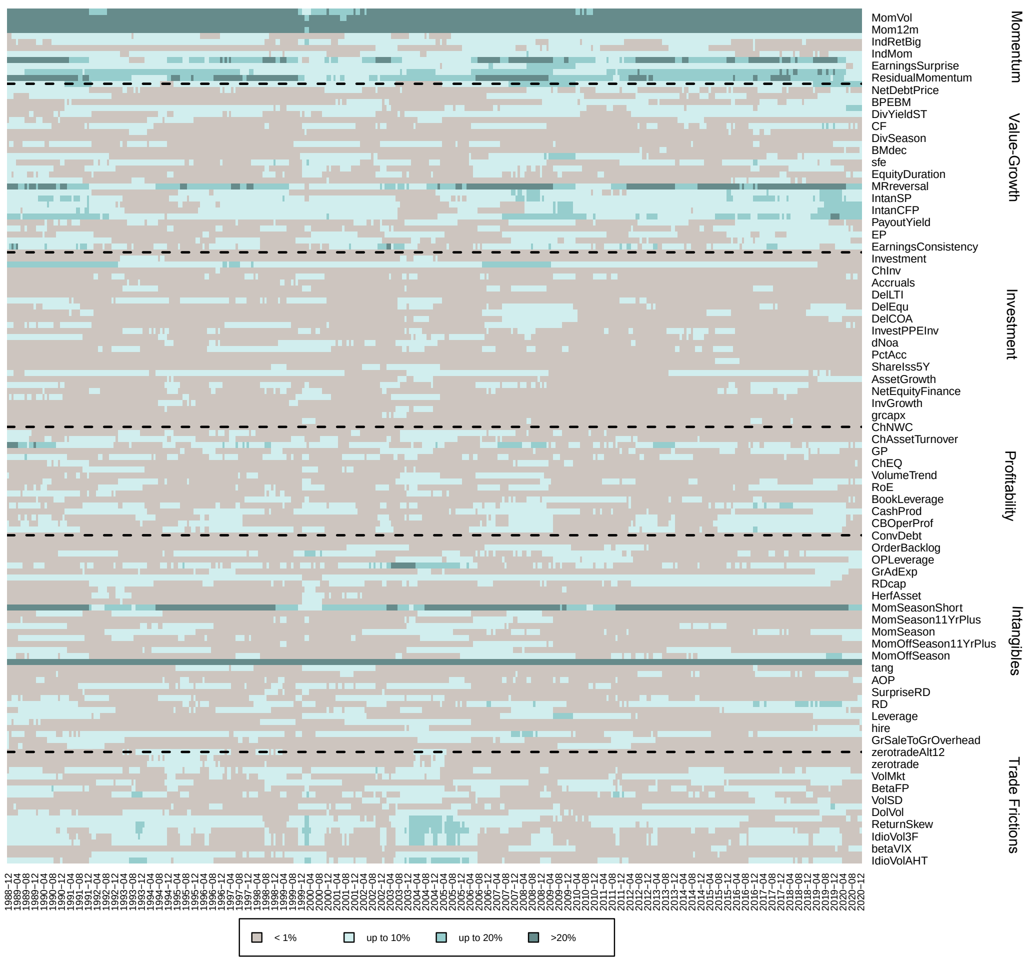

Figure 5: **Total variance decomposition - constant case**. The figure shows the total variance decomposition of each model as the sum of the contribution of anomalies (red), betas (green) and residual (gray) components. The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section OA.5.2. The analysis uses balanced panels over fixed-time windows of three years, with a reference period ranging from January 1986 to December 2020.

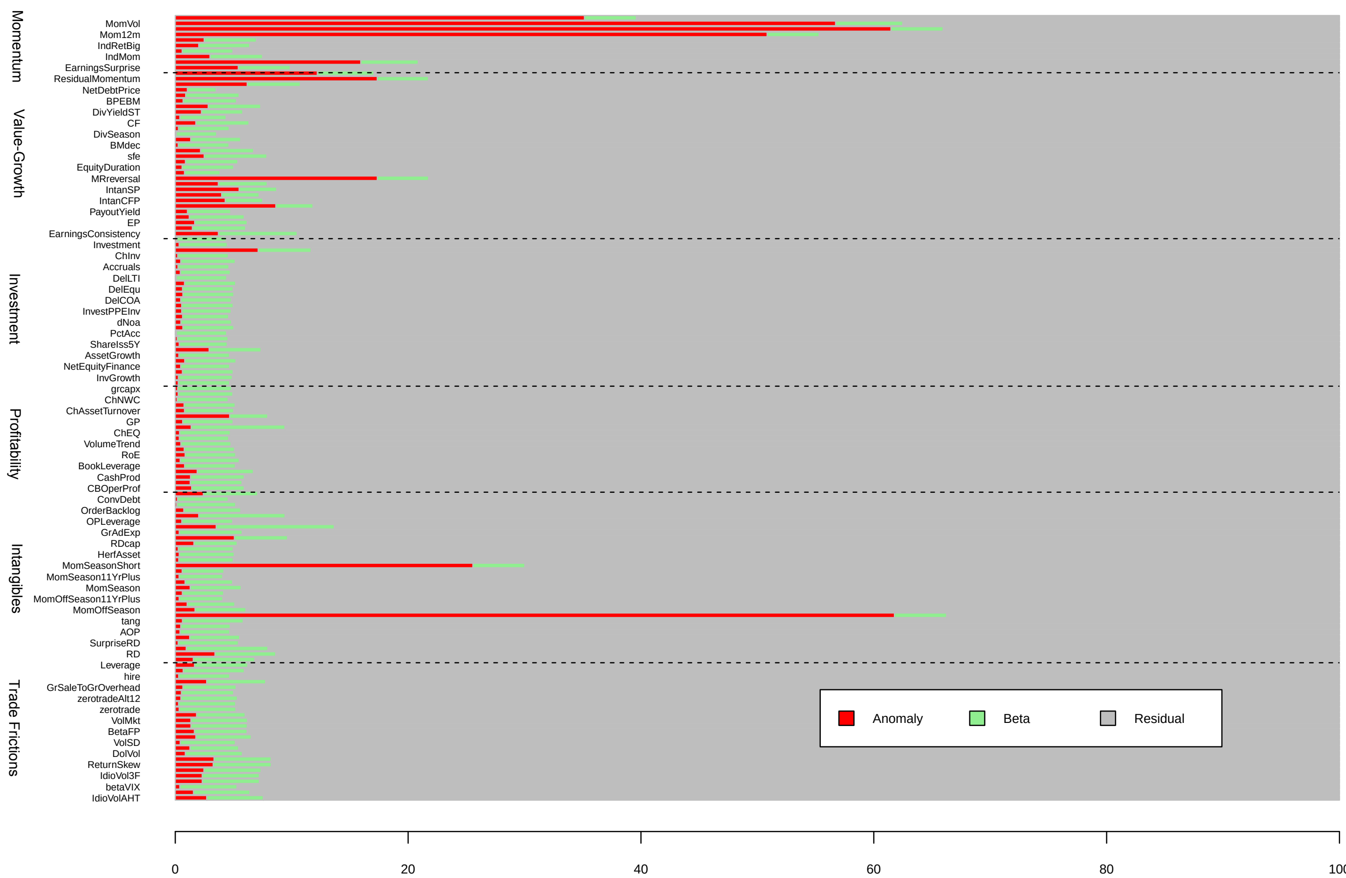

Figure 6: **Heatmap of the t-statistics distribution - time varying case**. The figure shows the heatmap of the t-statistics distribution obtained in each of the 170 univariate model (vertical axis) and for each month (horizontal axis). Each cell in the map represents the degree of statistical significance of the t-statistics with a different color, from gray (non-significant t-stat), to yellow (significance at only 10% level), orange (significance at 5% level), and red (significance at 1% level). The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section 5. of time-varying premia and anomalies. The analysis uses balanced panels at each month, with a reference period ranging from January 1986 to December 2020. At each month t, the market beta is obtained by running a first-pass regression using a rolling window on the past two years of data ($T = 24$).

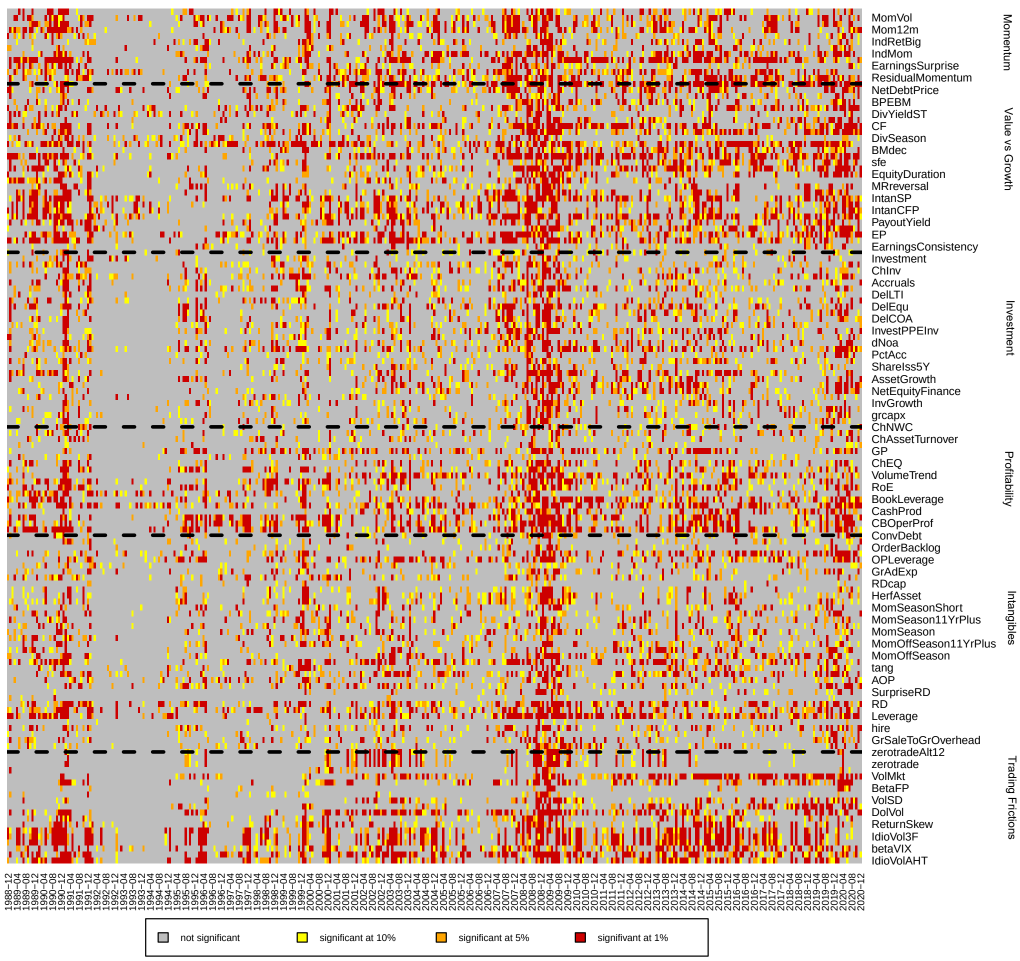

Figure 7: **Anomalies and financial crises - time varying case**. The figure shows the percentage of anomalies found to be significant at 5% (or lower) confidence level at each point in time. The light gray bands correspond to NBER recession dates and to various economic and financial crises. The results are obtained by performing univariate regressions of asset returns on the market factor and each of the 170 anomalies, using the theoretical results of Section 5. of time-varying premia and anomalies. The analysis uses balanced panels for each month, with a reference period ranging from January 1986 to December 2020. At each month t, the market beta is obtained by running a first-pass regression using a rolling window on the past two years of data ($T = 24$).

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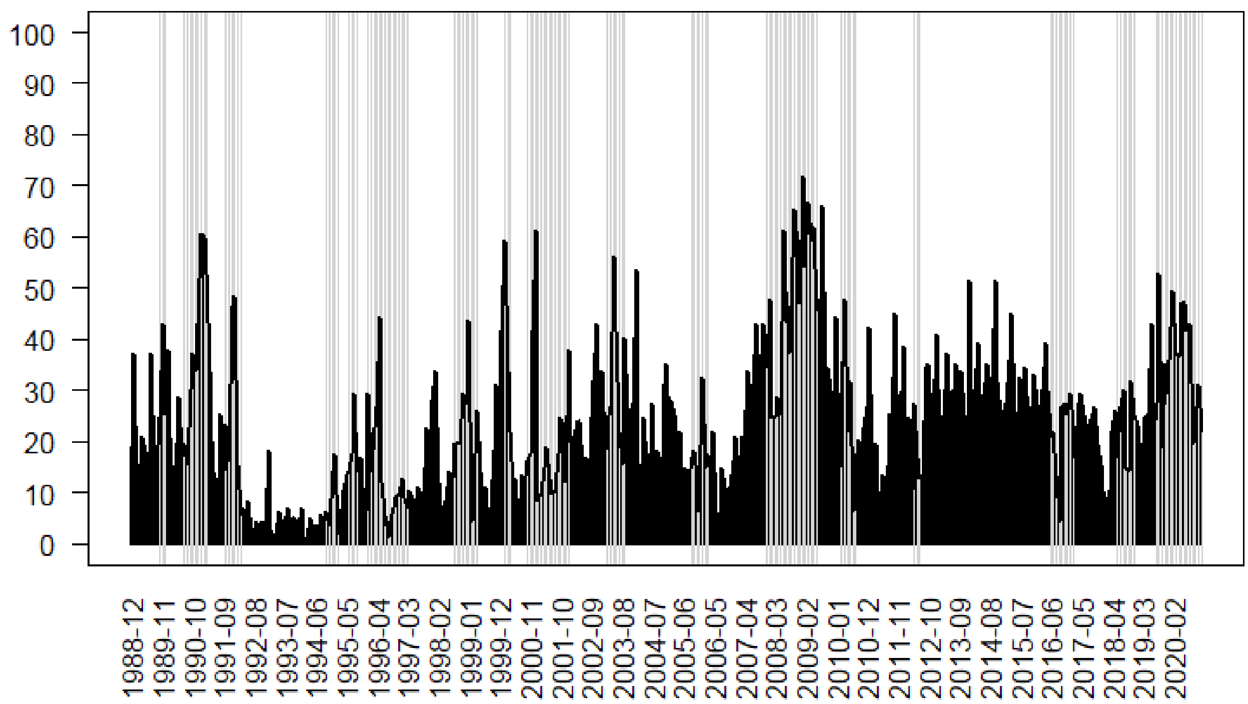


Figure 8: **"Best" representative sets of anomalies in multivariate regressions**. The figure shows the time-varying sets of anomalies that have been used to run multivariate regressions at each month. Each red point denotes the anomaly that has been picked in each category (vertical axis) and in each month (horizontal axis), using the empirical procedure described in Section 9.3.2.

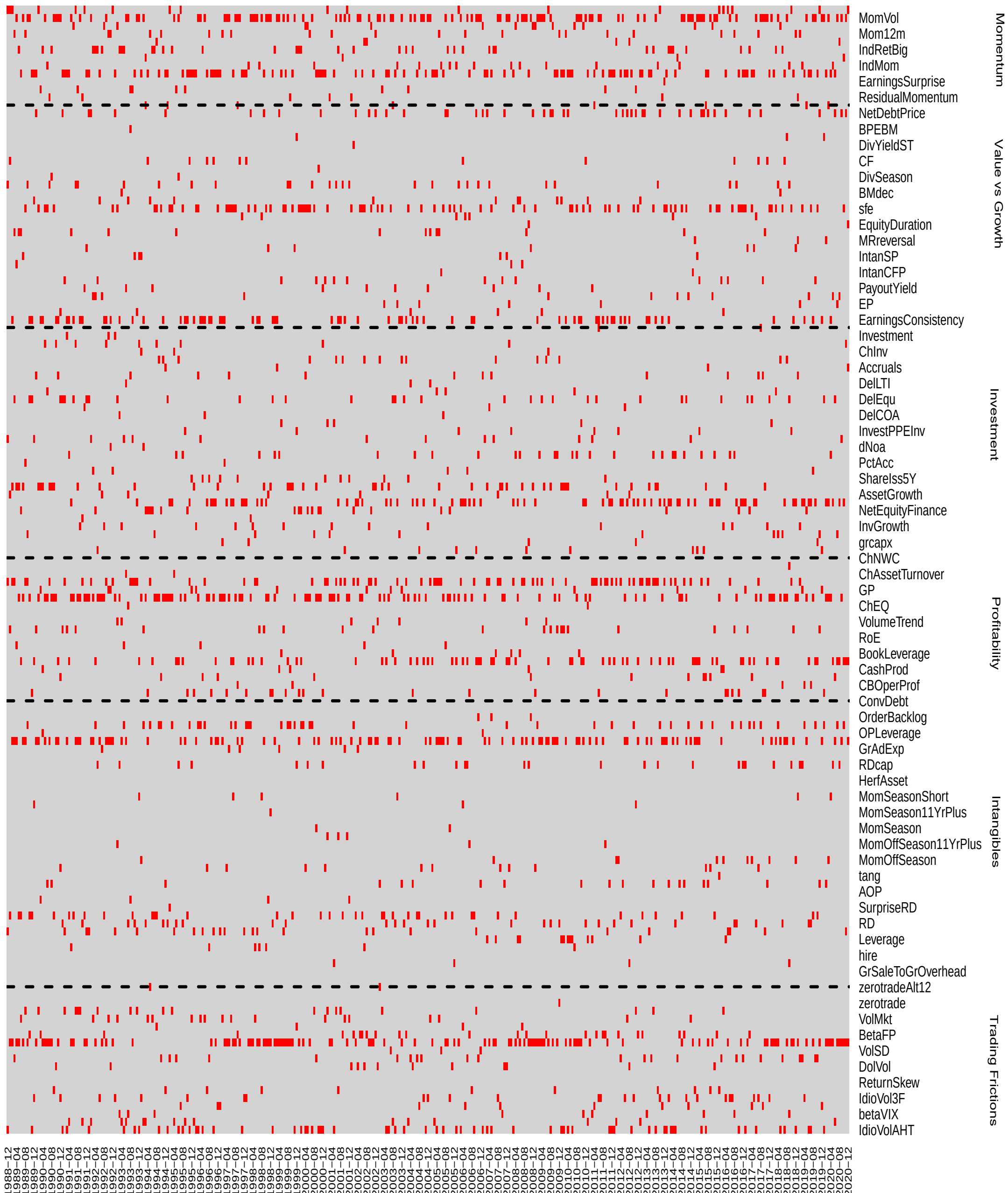
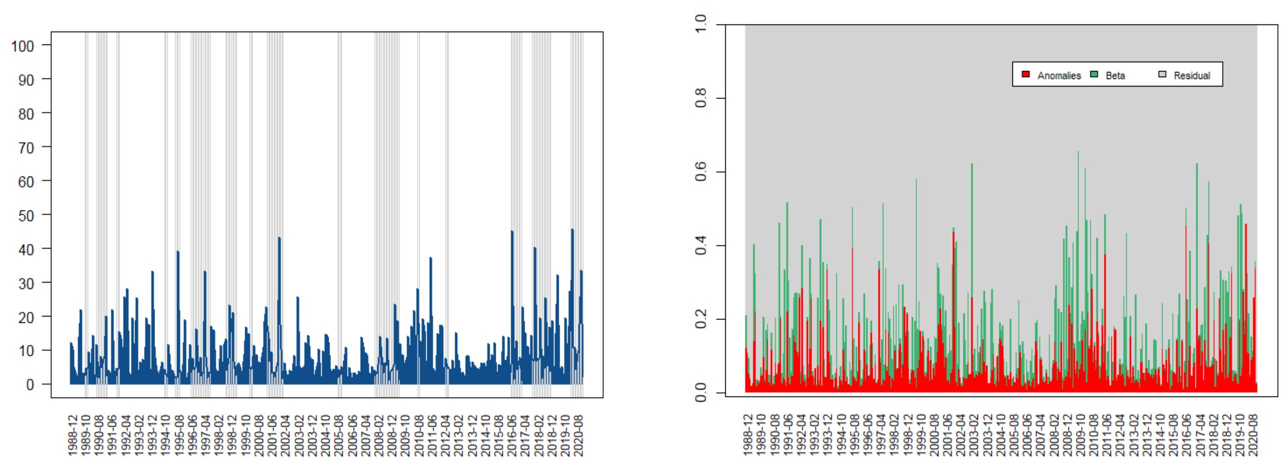


Figure 9: **Anomalies' contribution using time-varying multivariate regressions**. The figure shows the time series of the R_z^2 statistics (a), together with the total variance decomposition (b) obtained in each multivariate regression. The results are obtained by performing multivariate regressions of asset returns on the market factor and a set of six anomalies, selected using the empirical procedure described in Section 9.3.2. The analysis is based on the theoretical results of Sections 5 and 8 for time-varying premia and anomalies. The application uses balanced panels at each month, with a reference period ranging from January 1986 to December 2020. At each month t, the market beta is obtained by running a first-pass regression using a rolling window on the past two years of data ($T = 24$).



(a) Time series of R_z^2

(b) Total variance decomposition

Figure 10: **Testing the joint contribution of anomalies: time series of p-values**. The figure shows the time series of p-values associated with the $\mathcal{T}_z$ statistics for each multivariate model at each point in time. The null hypothesis is that $H_0 : \boldsymbol{\gamma}_z = \mathbf{0}_{K_z}$, against the alternative that at least one anomaly is different from zero, i.e., $H_1 : \boldsymbol{\gamma}_z \neq \mathbf{0}_{K_z}$, with $K_z = 6$. The yellow bands represent the p-values > 0.05 , for which we cannot find evidence to reject the null hypothesis. The blue lines refer to the p-values ≤ 0.05, i.e. all the periods in which we can reject the null hypothesis at the 5% confidence level. The analysis is based on the theoretical results of Theorem 6 (i), where the asymptotic distribution of the statistic $\mathcal{T}_z$ under H_0 has been tabulated using 10,000 random draws from six i.i.d. χ^2_1, weighted with the estimated values $(\hat{c}_1, ..., \hat{c}_6)$ obtained in each multivariate model. The results are obtained by performing multivariate regressions of asset returns on the market factor and a set of six anomalies, selected using the empirical procedure described in Section 9.3.2. The application uses balanced panels at each month, with a reference period ranging from January 1986 to December 2020. At each month t, the market beta is obtained by running a first-pass regression using a rolling window on the past two years of data ($T = 24$).